A Two Dimensional Infection Age – Structured Mathematical Model Of The Dynamics Of HIV/AIDS.

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Abstract

The paper considers two compartmentalized mathematical model of HIV/AIDS disease dynamics of the susceptible and infected members with age structure introduced in the latter class. The susceptibles are virus free but prone to infection through specific transmission pattern that is, coming in contact with infected body fluids such as blood, sexual fluids and breast milk. The total population is partitioned into two distinct classes giving rise to a set of model equations with one ordinary differential equation and one partial differential equation. Parameter values were used to represent the consequential interactive characteristics of the population. The equilibrium states and the corresponding characteristics equation were obtained. The Bellman and Cooke’s theorem is applied to analysed the equilibrium states of the model for stability and critical values of these parameters obtained. The result revealed that to sustain the population, the birth rate must be greater than the death rate among others.

1. Introduction

According to Benyah [3], mathematical modeling is an evolving process, as new insight is gained the process begins again as additional factors are considered. The research work proposes a deterministic mathematical model of HIV/AIDS disease dynamics resulting into a system of ordinary and partial differential equations. Differential equations form very important mathematical tools used in producing models of physical and biological processes. Burghes and Wood [4] opines that “…it could even be claimed that the spread of modern industrial civilization, for better or for worse, is partly a result of man’s ability to solve the differential equations which govern so many of our industrial processes, be them chemical or engineering”.

In this work, the population is partitioned into two compartments of the susceptible S(t), which is the class of members that are virus – free but are prone to infection as they interact with the infected class.

The infected class I(t) consists of members that contracted the virus and are at various stages of infection. This class is structured by the infection age, with the density function \( p(t, a) \) where ‘t’ is the time and ‘a’ is the infection age.

There is a maximum infection age \( T \) at which a member of the infected class must leave the compartment via death i.e. when \( a = T \) for \( 0 \leq a \leq T \). However, a member of the class could still die by natural causes at a rate \( \mu \), which is also applicable to the susceptible class S(t).

Members of S(t) move into I(t) at a rate \( \alpha \) due to negative change in behavior. The gross death rate via infection is given by \( \sigma(a) = \mu + \delta \tan \left( \frac{na}{2TK} \right) \). \( \delta \) is additional burden from infection while \( K \) is a control parameter associated with the measure of slowing down the death of the infected member, such as the effectiveness of the anti – retroviral drugs which give the victims longer life – span. A high rate of ‘K’ will imply high effectiveness of such measure and vice – versa.

It is assumed that while the new births in S(t) are born there in, the off-springs of I(t) are divided between S(t) and I(t) in the proportion \( \theta \) and 1 – \( \theta \) respectively that is, 1 – \( \theta \) of the off-springs of I(t) are born with the virus.
2. The model equations

\[ S^1 = (\beta - \mu) S(t) + \theta \beta I(t) - \alpha S(t) I(t) \] (1)

and \( I(t) = \int_0^T \rho(t, a) da, \) \( 0 \leq a \leq T \) (2)

\[ \frac{\partial \rho(t, a)}{\partial t} + \frac{\partial \rho(t, a)}{\partial a} + (t + \sigma(a)) \rho(t, a) = 0 \] (3)

Where \( \sigma(a) = \mu + \delta \tan \frac{\pi a}{TK} \) (4)

\[ S(0) = S_0, \quad I(0) = I_0 \] (7)

With the parameters given by

\( \beta = \text{natural birth rate for the population}; \)

\( \mu = \text{natural death rate for the population}. \)

\( \alpha = \text{rate of contracting the HIV virus}. \)

\( \sigma(a) = \text{gross death rate of the infected class}. \)

\( \delta = \text{additional burden from infection}. \)

\( K = \text{measure of the effectiveness of efforts at slowing down the death of infected members}. \)

\( \theta = \text{the proportion of the off-springs of the infected which are virus free at birth } 0 \leq \theta \leq 1. \)

\( T = \text{maximum infection age i.e. when } a = T \text{ the infected member dies of the disease}. \)

3. Equilibrium states

At the equilibrium states, let

\[ S(0) = x, \quad I(0) = y \] (8)

if \( \rho(t, a) = \varphi(a) \) (9)

from (1.2), \( y = \int_0^T \varphi(a) \) (10)

from (5), \( \varphi(0) = \beta(0) = \alpha xy + (1-\theta) \beta y \) (11)

Substituting (9) to (11) into (1) and (3)

\[ (\beta - \mu) x + \theta \beta y - \alpha xy = 0 \] (12)

3. The characteristics equation

As in Akinwande [1], let the equilibrium state be perturbed as follows:

\[ S(t) = x + p(t), \quad p(t) = \bar{p} e^{\lambda t} \] (22)

\[ I(t) = y + q(t); \quad q(t) = \bar{q} e^{\lambda t} \] (23)

Let \( \rho(t, a) = \varphi(a) + \eta a e^{\lambda t} \) (24)

With \( \bar{q} = \int_0^T \eta(a) da \) (25)

Substituting (22) to (25) into the model equations (1) and (3)
\( \lambda ) \frac{d}{dt} (x + p e^{\lambda t}) = (\beta - \mu) (x + p e^{\lambda t}) + \theta \beta (y + q e^{\lambda t}) - \alpha (x + p e^{\lambda t})(y + q e^{\lambda t}) \\
\lambda p e^{\lambda t} = (\beta - \mu) x + (\beta - \mu) p e^{\lambda t} + \theta \beta y + \theta \beta q e^{\lambda t} - \alpha \chi y - \alpha \chi q e^{\lambda t} - \alpha y p e^{\lambda t} - \alpha y p e^{\lambda t} 
\]

From equation (12) and neglecting terms of order 2, we have:

\( \lambda p e^{\lambda t} = (\beta - \mu) p e^{\lambda t} + \theta \beta q e^{\lambda t} - \alpha \chi q e^{\lambda t} - \alpha y q e^{\lambda t} \)

or

\( (\beta - \alpha \chi - \lambda) \theta + (\beta - \alpha x) \theta q = 0 \)  

Substituting (24) into (3), gives

\[
\frac{dp}{dt}(\phi(0) + \eta(0)e^{\lambda t}) + \frac{d}{da}(\phi(0) + \eta(0)e^{\lambda t}) + \sigma(a)(\phi(0) + \eta(0)e^{\lambda t}) = 0
\]

That is,

\[
\lambda \eta(0)e^{\lambda t} + \frac{d\eta(0)}{da} + e^{\lambda t} \frac{d\eta(0)}{da} + \sigma(a)\phi(0) + \sigma(a)\eta(0)e^{\lambda t} = 0
\]

Since

\[
\frac{d\eta(0)}{da} + \sigma(a)\phi(0) = 0
\]

Then

\[
\lambda (a)e^{\lambda t} + e^{\lambda t} \frac{d\eta(0)}{da} + \sigma(a)\eta(0)e^{\lambda t} = 0
\]

\[
\frac{d}{da} \eta(e^{\lambda t}) + (\sigma(a) + \lambda) \eta(0) = 0
\]

Solving the Ordinary Differentiated Equation (27), gives

\[
\frac{d\eta(0)}{\eta(0)} = -(\sigma(a) + \lambda)da
\]

\[
\eta(0) = \eta(0) e^{\int \sigma(a) + \lambda)da}
\]

(29)

Integrating (29) over \([0, T]\) gives

\[
\bar{q} = \eta(0) \int_{0}^{T} \left\{ e^{\int_{0}^{\sigma(a) + \lambda)da} ds} \right\} da
\]

or

\[
\bar{q} = \eta(0) b(\lambda)
\]

(30)

Since

\[
\bar{q} = \eta(0) b(\lambda)
\]

where

\[
b(\lambda) = \int_{0}^{T} \left\{ e^{\int_{0}^{\sigma(a) + \lambda)da} ds} \right\} da
\]

(31)

\[\eta(0)\] is calculated as follows:

From (11), \( \phi(0) = \alpha xy + (1-\theta)\beta y \).

\[\text{and } \phi(t, 0) = \beta(t) = \phi(0) + \eta(0)e^{\lambda t} \]

(32)

From (5), \( \beta(t) = \alpha s(t) + (1-\theta)\beta l(t) \)

Substituting (22) into (25) into (5) and using (11) and (32)

\[
\beta(t) = \alpha(x + pe^{\lambda t})(y + qe^{\lambda t}) + (1-\theta)\beta(y + qe^{\lambda t})
\]

\[
= \alpha xy + \alpha ype^{\lambda t} + \alpha xqe^{\lambda t} + \alpha ye^{\lambda t} + (1-\theta)\beta y + (1-\theta)\beta qe^{\lambda t}
\]

(33)

Compare this with (32) using (11) for \( \phi(0) \) gives

\[
\alpha xy + (1-\theta)\beta y + \eta(0)e^{\lambda t} = \alpha xy + \alpha ype^{\lambda t} + \alpha xqe^{\lambda t}
\]

\[
+ \alpha ye^{\lambda t} + (1-\theta)\beta y + (1-\theta)\beta qe^{\lambda t}
\]

neglecting terms of order 2

\[
\eta(0) = \alpha \beta y + \alpha \beta x + (1-\theta)\beta q
\]

(34)

Substituting \( \eta(0) \) in (30)

\[
\bar{q} = (\alpha y\beta + \alpha xq + (1-\theta)\beta q) b(\lambda)
\]

(35)

\[
\alpha y\beta + [(\alpha x + (1-\theta)\beta) b(\lambda) - 1] \bar{q} = 0
\]

(36)

Using (26) and (36), we obtain the Jacobian determinant for the system with the eigen value \( \lambda \)

\[
\begin{vmatrix}
\beta - \mu - \alpha x - \lambda & \theta \beta - \alpha x \\
\alpha y & (\alpha x + (1-\theta)\beta) b(\lambda) - 1
\end{vmatrix}
\]

= 0

(37)

and the characteristics equation is given by:

\[
(\beta - \mu - \alpha x - \lambda)[(\alpha x + (1-\theta)\beta) b(\lambda) - 1] - \alpha y(\theta \beta - \alpha x) = 0
\]

(38)
5. Stability of the zero equilibrium state

At the zero equilibrium state \((x, y) = (0, 0)\), the characteristic equation becomes:

\[
(\beta - \mu - \lambda)[(1-\theta)b(\lambda)-1] = 0 \quad (39)
\]

That is,

\[
\lambda_1 = \beta - \mu \quad (40)
\]

This shows that \(\lambda_1 < 0\), if \(\beta < \mu\)

The nature of the roots of the transcendental equation \((1-\theta)b(\lambda)-1\) is now investigated.

Since \(b(\lambda) = \int_0^T e^{-\lambda a} \pi(a) \, da\), which implies that

\[
b(\lambda) = \int_0^T e^{-\lambda a} \pi(a) \, da \quad (42)
\]

Using the approximation

\[
b(\lambda) = \int_0^T (1-\lambda a) \pi(a) \, da = \int_0^T \pi(a) \, da - \lambda \int_0^T a \pi(a) \, da \quad (43)
\]

\[
= \bar{\pi} - \lambda \bar{A} \text{ where } \bar{A} = \int_0^T a \pi(a) \, da
\]

So, \((1-\theta)b(\lambda)-1 = 0\) takes the form:

\[
(1-\theta)b(\bar{\pi} - \lambda A)-1 = 0 \quad (44)
\]

\[
\lambda = \frac{(1-\theta)b(\bar{\pi}) - 1}{(1-\theta)bA} \quad (45)
\]

So, \(\text{sign } \lambda = \text{sign } \{(1-\theta)\bar{\pi} - 1\} \quad (46)

Let \(D_1(k) = (1-\theta)\bar{\pi} - 1 \quad (47)

So, the origin will be stable when \(D_1(k) < 0\) and Unstable when otherwise.

From table 1, \(D_1(k) > 0\) when \(\beta > \mu\), which implies the instability of the origin.

6. Stability analysis of the non zero state

At the non zero state \((x, y) = (0, 0)\), the characteristic equation (38) taking it in the form

\[
H(\lambda) = \frac{1}{\infty} (1 - (1-\theta)b\bar{\pi}) - \lambda \frac{(\beta - \mu)\frac{1}{\infty} [1-\frac{1}{(1-\theta)b\bar{\pi}}]}{[1-\lambda(1-\theta)b\bar{\pi} + \theta\beta]} \quad (48)
\]

To analyse the non – zero state for stability, we shall apply the result of Bellman and Cooke [2] to the characteristics equation (38) taking it in the form

\[
H(\lambda) = 0
\]

if we set \(\lambda = iw\); and have \(H(iw) = F(w) + iG(w)\) (49)

Since \(b(\lambda) = \int_0^T \exp \{-\int_0^a (\lambda + \sigma(s))ds\} da\)

\[
b(iw) = \int_0^T \exp \{-\int_0^a (iw + \sigma(s))ds\} da = \int_0^a e^{-iw a} \bar{\pi}(a) da \quad (50)
\]

\[
f(w) = \int_0^T \pi(a) \cos wa \quad (52)
\]

\[
g(w) = -\int_0^T \pi(a) \sin wa \quad (53)
\]
so, \( f(0) = \int_0^a \pi(a) da = \bar{\pi} \)  \( (54) \)

\[ g(0) = 0 \]  \( (55) \)

Also, \( f(w) = -\int_0^a \pi(a) \sin(wa) da \)

\[ f(0) = 0 \]  \( (56) \)

and, \( g(w) = -\int_0^a \pi(a) \cos wa da \)

\[ g(0) = -\int_0^a \pi(a) da = -A \]  \( (57) \)

Thus, \( H(iw) = (\beta - \mu - \alpha y - iw) [(\alpha x + (1 - \theta )\beta) - \alpha y (\theta \beta - \alpha x)] \)  \( (58) \)

\[ H(iw) = (\beta - \mu - \alpha y - iw) [(\alpha x + (1 - \theta )\beta) f(w) + ig(w) - 1] - \alpha y (\theta \beta - \alpha x) \]

\[ F(w) = (\beta - \mu - \alpha y) (\alpha x + (1 - \theta )\beta) f(w) + w(\alpha x + (1 - \theta )\beta)ig(w) - (\beta - \mu - \alpha y) - \alpha y (\theta \beta - \alpha x) \]  \( (59) \)

\[ G(w) = (\beta - \mu - \alpha y) (\alpha x + (1 - \theta )\beta) g(w) - w (\alpha x + (1 - \theta )\beta) f(w) + w \]  \( (60) \)

\[ G(\theta ) = 0 \]

\[ F(\theta ) = (\beta - \mu - \alpha y) (\alpha x + (1 - \theta )\beta) \bar{\pi} - (\beta - \mu - \alpha y) - \alpha y (\theta \beta - \alpha x) \]  \( (61) \) and

\[ G^1(\theta ) = - (\beta - \mu - \alpha y) (\alpha x + (1 - \theta )\beta) \bar{\pi} + 1 \]  \( (62) \)

\[ F^1(\theta ) = 0 \]

For stability or otherwise of the equilibrium state, we need to satisfy the condition for which the inequality \( F(\theta ) G^1(\theta ) - F^1(\theta ) G(\theta ) > 0 \) holds.

The inequality then gives \( F(\theta ) G^1(\theta ) > 0 \)

Let \( D_2(\theta ) = F(\theta ) G^1(\theta ) \)

Then, the non-zero state will be stable when \( D_2(\theta ) > 0 \)

\( (63) \)

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**Table 2. Stability analysis of the non-zero state.**

<table>
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<tr>
<th>K</th>
<th>( \beta )</th>
<th>( \delta )</th>
<th>( \mu )</th>
<th>( \alpha )</th>
<th>( \theta )</th>
<th>T</th>
<th>( D_2(\theta ) )</th>
<th>Remark</th>
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From table 2, \( D_2(\theta ) > 0 \) which implies the stability of the non zero state.
Conclusion

The zero equilibrium state which is the state of population extinction will be stable when the birth rate is less than the death rate in addition to meeting the requirement of inequality $D_1(k)<0$. The non zero state, which is the state of population sustenance will be stable if the inequality (63) is satisfied. So efforts must be geared toward meeting this non zero stability bound through public enlightenment.

References


