A Transfer Function Model of Thrust Dynamics for Multi-Rotor Helicopters

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Abstract—This paper proposes a transfer function model for a dynamic thrust of a motor-driven propeller widely used for small multi-rotor helicopters. From frequency responses of a propeller speed with respect to a PWM (pulse width modulation) duty ratio of a motor speed controller, it is found that a thrust dynamics can be accurately modelled as a first-order transfer function. This frequency-domain result is compared to a time-domain method which is based on a step response of a thrust.

Keywords—Thrust Dynamics, Transfer Function, Multi-rotor Helicopter

I. INTRODUCTION

Obtaining a reliable mathematical model of a thrust force is one of the most important and difficult steps in a controller synthesis for a multi-rotor helicopter.

There are fundamental difficulties in modeling a dynamic thrust of a propeller, as addressed in [1]. First, a commercial ESC (electronic speed controller) driving a BLDC (brushless direct current) motor itself is a microprocessor-based digital system and as such it has its own operational parameters but information on those parameters is usually undisclosed. Furthermore, with a custom firmware for an ESC, those parameters and other settings can be also modified. Second, an aerodynamic relation between a propeller speed and a thrust is highly nonlinear and complicated. Furthermore, depending on an operating condition, the relation can be significantly changed.

Because of the aforementioned difficulties, for a reliable characterization of a thrust force, it is generally unavoidable to depend on experimental procedures. This situation motivated us to develop a simple experimental method for a characterization of both a static and dynamic thrust in [2]. The proposed experimental procedure of [2] uses a load-cell type force sensor for a thrust measurement and an optical sensor for a speed measurement. The manual procedure performed in [2] for a static thrust is completely automatized with a microprocessor-based thrust measurement system in [3].

However an identification of a dynamic thrust is much harder compared to that of a static thrust from the following reason. A load-cell sensor for a thrust measurement uses a high-gain analog instrument amplifier. This high-gain amplification combined with a continual switching of large currents in a BLDC motor which needs to be installed near the load-cell (see Fig. 1), make a thrust sensor signal extremely vulnerable to electrical noises. This noise is significant and cannot be easily removed especially when a propeller speed changes abruptly. As a consequence of this, a precise measurement of a dynamic thrust is challenging in general.

Nevertheless, we have obtained a rough characterization of a dynamic thrust in [2], assuming that the dynamic relation between an ESC PWM command and a thrust can be described as a first-order transfer function. Parameters of the first-order transfer function were estimated from a step response of a thrust with respect to a step PWM command in [2].

However a fundamental limitation in the dynamic thrust model of [2] is that the first-order dynamic relation between a thrust and an ESC command is presumed, without a sufficient justification. In fact the same first-order model was also adopted in [4] for an example.

A key contribution of this paper is to experimentally substantiate the presumed first-order dynamics. To be concrete, we obtained a frequency response between an ESC command and a thrust force, and confirmed that the dynamic thrust can be rather precisely modelled as a first-order system. Furthermore, our identification method based on a frequency response was compared to the time-domain counterpart in [2].

II. DYNAMIC MODELING OF THRUST

A. Experimental Setup

Fig.1 from [3] shows our thrust experiment system composed of an ESC, a BLDC motor, a load-cell for a thrust measurement and an optical sensor for a speed measurement. Components of our thrust measurement system in Fig.1 have the technical specifications in Table 1, cited from [1].

![Figure 1 Sensor Configuration [3]](image-url)
TABLE I. COMPONENTS SPECIFICATION [1]

<table>
<thead>
<tr>
<th>Component</th>
<th>Specification</th>
</tr>
</thead>
<tbody>
<tr>
<td>BLDC Motor</td>
<td></td>
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<tr>
<td>Motor Outer Diameter</td>
<td>58.5 mm</td>
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<tr>
<td>Stator Diameter</td>
<td>50.0 mm</td>
</tr>
<tr>
<td>Speed per Volt</td>
<td>340 RPM/V</td>
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<tr>
<td>Stator Number</td>
<td>12</td>
</tr>
<tr>
<td>Motor Poles</td>
<td>14</td>
</tr>
<tr>
<td>Weight</td>
<td>168 g</td>
</tr>
<tr>
<td>Propeller</td>
<td></td>
</tr>
<tr>
<td>Length</td>
<td>18 inches</td>
</tr>
<tr>
<td>Pitch</td>
<td>5.5 inches</td>
</tr>
<tr>
<td>Material</td>
<td>carbon fiber</td>
</tr>
<tr>
<td>Blade Root Thickness</td>
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</tr>
<tr>
<td>Load Cell</td>
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</tr>
<tr>
<td>Capacity</td>
<td>5 kg</td>
</tr>
<tr>
<td>Resistance</td>
<td>1000 Ω</td>
</tr>
<tr>
<td>Material</td>
<td>Aluminum</td>
</tr>
<tr>
<td>Nonlinearity</td>
<td>0.05 %</td>
</tr>
<tr>
<td>ESC</td>
<td></td>
</tr>
<tr>
<td>Output (continuous)</td>
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</tr>
<tr>
<td>Battery</td>
<td></td>
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<tr>
<td>Capacity</td>
<td>10000 mAh</td>
</tr>
<tr>
<td>Nominal Voltage</td>
<td>22.2 V</td>
</tr>
<tr>
<td>Discharging Rate</td>
<td>25C</td>
</tr>
</tbody>
</table>

B. Time-Domain Method

In this section we will apply the time-domain identification method for a dynamic thrust proposed in [2], to our case study model.

Our ESC has a custom firmware provided by BLheliSuite 14.2.0.1 [5]. This firmware allows us to use a 4 kHz PWM signal as a command signal. Details on this custom firmware and its driving signals can be found in [2].

From experiments, correlations between a propeller angular speed, a static thrust force and a PWM duty ratio were found to be as shown in Fig. 2-4. In addition, the step-response of a thrust with respect to step PWM command given in Fig. 5 is cited from [1]. The data with a label “4kHz” is to be used.

It is generally accepted that the relation between a propeller speed and a thrust is static. This static relation in our case could be found from a quadratic interpolation of the data in Fig. 4 as

\[ T = 1.18 \times 10^{-6} \omega^2 \]  

where \( T \) denote the thrust in Newton and \( \omega \) denotes the speed in RPM unit.

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Figure 2 Speed versus Duty Ration

Figure 3 Thrust versus Duty Ratio

Figure 4 Thrust versus Speed

Figure 5 Thrust Step Response [1]
From the static relation (1), in principle, we have only to identify the dynamic response of either a thrust or a speed with respect to a PWM command. However the measurement of a speed is much more robust to electrical noises, compared to that of a thrust, and therefore we will investigate the dynamic property of a propeller speed first.

As a first step, we chose an operating point of the duty ratio of an ESC command input as \( d_{nom} \approx 30 \% \). Then from Fig. 2 the corresponding operating point of a propeller speed is around \( \omega_{nom} = 2400 \) rpm.

From a linearization
\[
d = d_{nom} + \Delta d, \quad \omega = \omega_{nom} + \Delta \omega
\]
we need to identify two unknown parameters \((k, \tau)\) in the first-order transfer function
\[
\frac{\Delta \omega(s)}{\Delta d(s)} = \frac{k}{s + \frac{1}{\tau}}
\]
where \( \Delta \omega(s), \Delta d(s) \) denote the Laplace transform of the input \( \Delta d(t) \) and the output \( \Delta \omega(t) \), respectively.

In (2), the unknown parameter \( \tau \) can be determined from the step response shown in Fig. 5. Specifically, by reading a rising time 0.16 (sec) from Fig. 5, we obtain \( \tau = 0.16 \) sec.

As a second step, for the unknown parameter \( k \), we note that the DC (static) gain
\[
\frac{\Delta \omega(0)}{\Delta d(0)} = \frac{k}{1/\tau} = k \tau
\]
should be equal to the slope 60.8 (rpm/duty (%)) of a tangential line at \( d_{nom} = 30 \% \) as illustrated in Fig. 2. This gives \( k = \frac{60.8}{\tau} = 380.0 \).

In summary, around an operating point \((d_{nom}, \omega_{nom}) = (30, 2400)\), the transfer function between a duty-ratio command and a propeller speed is given as
\[
\frac{\Delta \omega(s)}{\Delta d(s)} = \frac{380.0}{s + 6.67} (\text{ RPM } / \%)
\]
where \( \omega_{nom} = 2400 \) rpm.

Note that at the operating point the thrust force \((1)\) can be linearized as
\[
T = T_{nom} + \Delta T = \alpha \omega_{nom}^2 + 2 \omega_{nom} \Delta \omega = 6.80 + 0.0057 \Delta \omega \quad (\alpha = 1.18 \times 10^{-6})
\]
From this result, the transfer function between a duty command and a thrust at an operating point \((d_{nom}, T_{nom}) = (30, 6.80)\) is given
\[
\frac{\Delta T(s)}{\Delta d(s)} = \frac{2.17}{s + 6.67} \left( \frac{N}{\%} \right).
\]
We note that in our experimental data in Fig. 4, the nominal thrust is around 6.34 (N) which is slightly different from the estimated value 6.80 in (4).

C. Frequency Response Method

In this section, without resorting to the assumption that the transfer function from a PWM duty-ratio command to a thrust is a first-order system, we measure the frequency response of a thrust and characterize its transfer function representation.

The frequency response of a thrust, with respect to a PWM duty-ratio, was obtained with a dynamic signal analyzer Agilent 35670A [6].

As our signal analyzer is an analog device, it was necessary to convert an analog input signal from the signal analyzer to a digital PWM signal for an ESC, and a digital pulse train from a photo sensor for a speed measurement into an analog output signal whose voltage is proportional to a propeller speed. In addition, in order to handle negative voltages from the analyzer, an operational amplifier circuit was designed for a voltage shift. Furthermore, the zero voltage of both input and output signals of the analyzer were mapped to an operating point in the previous section.

The analog-digital conversion, digital-analog conversion and the mapping between voltage signals and an operating point were implemented with a microprocessor (Arduino Due ©)-based thrust measurement board described in [3]. A schematic diagram of various signal conditioning of our experiment are illustrated in Fig. 5.

Note that in Fig. 5 the ADC has a DC gain \( \frac{\Delta d}{u} = 10 (\% \text{ cm}) \) and the DAC has a gain \( \frac{V}{\Delta \omega} = \frac{1}{1000} \left( \frac{V}{\text{ RPM}} \right) \) where \( u(t) \) and \( y(t) \) denote the output (input) and input (output) of the signal analyzer (thrust system, respectively). As a result, the relation
\[
\frac{\Delta \omega(s)}{\Delta d(s)} = 100 \frac{Y(s)}{U(s)}
\]
holds where the transfer function \( \frac{Y(s)}{U(s)} \) corresponds to a frequency response that the signal analyzer will estimate.

The signal analyzer under the configuration of Fig. 6 gave the frequency response in Fig. 7. The frequency span was \([0.1, 10] \) (Hz) and frequency responses at 401 different frequencies were measured with a swiping sinuisoidal signal.

A critical implication from both the magnitude and phase plots in Fig. 7 is that, qualitatively, the transfer function between a PWM command and a propeller speed (and a thrust, too) is a typical first-order transfer function.

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**Diagram Reference:**

Figure 6 A Schematic Diagram

The analog-digital conversion, digital-analog conversion and the mapping between voltage signals and an operating point were implemented with a microprocessor (Arduino Due ©)-based thrust measurement board described in [3]. A schematic diagram of various signal conditioning of our experiment are illustrated in Fig. 5.
This frequency-domain observation justifies our previous assumption of the first-order transfer function.

For a quantitative analysis of the frequency response data, we made a comparison between the transfer function in (3) which is based on a step-response of a thrust, and the frequency response data. To be concrete, in Fig. 8, the dashed blue line is the Bode plot of the transfer function \( \frac{\Delta \omega(\tau)}{100 \Delta d(\tau)} \) and the solid red line is the experimental data in Fig. 7. The scaling factor \( \frac{1}{100} \) comes from the relation (6).

The two frequency responses in Fig. 8 show a surprising agreement in overall, even though our choice of the rising time \( \tau = 0.16 \) (sec) from Fig. 5 was not precise. This result also suggests that the time-domain approach in [2] for a thrust transfer function is reliable.

Finally, Fig. 9 shows an example of input-output signals of the signal analyzer during a frequency response experiment. The frequency span in this figure is around \([2.5, 5.5]\) (Hz).

### III. CONCLUSION

A frequency response of a dynamic thrust with respect to a PWM duty-ratio command for an electronic speed controller was obtained with a dynamic signal analyzer. It was found that the transfer function between a PWM command and a thrust can be precisely modelled as a first-order transfer function. This transfer function gives a good agreement with another transfer function estimated from a step response of a thrust.

### REFERENCES


