A Three Dimensional Infection Age-Structured Mathematical Model Of Hiv/Aids Dynamics Incorporating Removed Class

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Abstract

The research proposes a three dimensional infection age structured deterministic mathematical model of the HIV/AIDS disease dynamics and introduce requisite parameters on the effects of public campaign and drug administration aimed at slowing down the death of the victims. The population is divided into three compartments of the susceptible, removed and the infected members with age structure introduced in the infected class. The susceptible are virus free but are prone to infection through specific transmission pattern that is, coming in contact with infected body fluids such as blood, sexual fluids and breast milk. Members of the removed class are not prone to infection due to their adherence to warnings or change in behavior through public enlightenment. Members in the infected class are HIV virus carriers and are at various stages of infection. The three distinct classes led to a set of model equations with two ordinary differential equations and one partial differential equation with parameter values used to represent the consequential interactive characteristics of the population. The zero and non zero equilibrium states were analysed for stability or otherwise of the model and bounds obtained for sustenance of the population.

1. Introduction

The research work proposes a deterministic mathematical model of the Human Immunodeficiency Virus / Acquired Immune Deficiency Syndrome (HIV / AIDS) disease dynamics which is a system of ordinary and partial differential equations. The population p(t) is divided into three classes of the susceptible s(t); this is the class in which members are virus free but are prone to infection as they interact with the infected class; the next class is the removed R(t). which is the class of those not susceptible to infection, possibly due to their adherence to warnings or changed behavior or enlightenment through public campaign; while the last class is the infected I(t); this is the class of those that have contracted the virus and are at various stages of infection.

The infected class I(t) is structured by the infection age with the density function $\ell(t,a)$, where t is the time parameter and a is the infection age. There is a maximum infection age T at which a member of the infected class I(t) must quit the compartment via death; and so $0 \le a \le T$. This not withstanding a member of the class could still die by natural causes at rate μ, the latter is also applicable to the susceptible class s(t) and the removed class R(t).

Members of the class S(t) move into the class R(t) due to change in behavior or / and as a result of effective public campaign at a rate γ.

The death rate via infection is given by $\sigma(a) = \mu + \delta \tan \frac{\pi a}{2TK}$; δ is the maximum death rate due to infection while K is a control parameter which could be associated with the measure of slowing down the death of the infected, such as the effectiveness of the anti – retroviral drugs which give victims

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longer life span. A high value of K will imply high effectiveness of such measure and vice versa.

It is assumed that while the new births in S(t) are born therein, the off-springs of I(t) are divided between S(t) and I(t) in the proportions θ and (I - θ) i.e. a proportion (I - θ) of the offspring of I(t) are born with the virus.

The equilibrium states are then analysed for stability.The work complements and extends the work of Bawa [2] by incorporating the removed class. According to Benyah [4], mathematical modeling is an evolving process, as new insight is gained the process begins again as additional factors are considered.

2. The Model Equations

The model equations are given by:

$$\frac{ds(t)}{dt} = \beta\big(S(t) + R(t)\big) - (\gamma + \mu)S(t) + \theta\beta I(t) - \alpha S(t)I(t) \quad (1)$$

$$\frac{dR(t)}{dt} = \gamma s(t) - \mu R(t) \quad (2)$$

$$\frac{\partial \ell(t,a)}{\partial t} + \frac{\partial \ell(t,a)}{\partial a} + \big(\mu + \delta(a)\big)\ell(t,a) = 0 \quad (3)$$

$$\sigma(a) = \mu + \delta \tan\frac{\pi a}{2TK} \quad (4)$$

$$I(t) = \int_0^T \ell(t,a)da \qquad 0 \le a \le T \quad (5)$$

$$\ell(t,0) = B(t) = \alpha S(t)I(t) + (I-\theta)\beta I(t);\ \ell(0,a) = \phi(a) \quad (6)$$

With the parameters given by:

β = natural birth rate for the population

μ = natural death rate for the population

α = rate of contracting the HIV virus

σ(a) = gross death rate of the infected class.

δ = additional burden from infection

K = measure of the effectiveness of efforts at slowing down the death of infected members like drug administration.

γ = rate of removal of the susceptible into the removed class; due to public campaign; i.e. measure of the effectiveness of the public campaign against infection

θ = the proportion of the off-spring of the infected which are virus free at birth $0 \le \theta \le 1$

t = time

a = infection age

T = maximum infection age; i.e. when a = T, the infected member dies of the disease.

3. Equilibrium States

At the equilibrium state, let

$$S(t) = x;\ R(t) = y;\ I(t) = z;\ \ell(t,a) = \phi(a) \quad (7)$$

$$Z = \int_0^T \phi(a)da \quad (8)$$

$$\phi(0) = B(0) = \alpha xz + (1-\theta)\beta z \quad (9)$$

Substituting (7) and (8) into (1) and (3) give

$$\beta(x+y) - (\gamma+\mu)x + \theta\beta z - \alpha xz = 0 \quad (10)$$

$$\gamma x - \mu y = 0 \quad (11)$$

$$\frac{d\phi(a)}{da} + h(a)\phi(a) = 0 \quad (12)$$

Where $h(a) = \mu + \sigma(a) = \mu + \delta e^{-k(I-a)}$ (13)

From (12)

$$\phi(a) = \phi(0)\exp\{-\int_0^a \sigma(s)ds\} \quad (14)$$

Let $\pi(a) = \exp\{-\int_0^a \sigma(s)ds\}$ (15)

Equation (14) becomes

$$\phi(a) = \phi(0)\,\pi(a) \quad (16)$$

From equation (8)

$$z = \phi(0)\bar{\pi} \quad (17)$$

Where

$$\bar{\pi} = \int_a^T \pi(a)da \quad (18)$$

Using equation (9) in equation (17) gives

$$z = [\alpha xz + (1-\theta)\beta z]\bar{\pi} \quad (19)$$

We then solve equations (10), (11) and (19) simultaneously, for (x, y, z) which gives the non-zero equilibrium state of the model.

From equation (19)

$$(\alpha x + (1-\theta)\beta)\bar{\pi} = 1$$

So

$$x = \frac{1-(1-\theta)\beta\bar{\pi}}{\alpha\bar{\pi}} \qquad (20)$$

Substituting equation (20) in (11) give

$$y = \frac{\gamma(1-(1-\theta)\beta\bar{\pi})}{\alpha\mu\bar{\pi}} \qquad (21)$$

Substituting equations (21) and (20) in (10) give

$$\mu\beta - (1-\theta)\mu\beta^2\bar{\pi} + \beta\gamma - (1-\theta)\gamma\beta^2\bar{\pi} - \mu\gamma + (1-\theta)\mu\gamma\beta\bar{\pi} - \mu^2 + (1-\theta)$$

$$\mu^2\beta\bar{\pi} + \left(\theta\beta\alpha\mu\bar{\pi} - \mu\alpha(1-(1-\theta)\beta\bar{\pi})\right)z = 0$$

$$z = \frac{(\beta\mu+\beta\gamma-\mu^2-\mu\gamma)(1-(1-\theta)\beta\bar{\pi})}{\mu\alpha(1-(1-\theta)\beta\bar{\pi})-\theta\beta\alpha\mu\bar{\pi}}$$

$$z = \frac{(\beta-\mu)(\mu+\gamma)(1-(1-\theta)\beta\bar{\pi})}{\mu\alpha(1-(1-\theta)\beta\bar{\pi})-\theta\beta\bar{\pi}} \qquad (22)$$

Where

$$\bar{\pi} = \int_0^T \pi(a)da : \pi(a) = \exp\left\{\int_0^1 -h(s)ds\right\}$$

(23)

The equilibrium states are the zero state (x, y, z) = (0, 0, 0) and the non-zero state given by the equations (20) to (22).

4 .The Characteristic Equation

As in Akinwande [1], the equilibrium states are perturbed as follow:

Let $\mathrm{S(t)} = x + \mathrm{p(t)}; \mathrm{p(t)} = \bar{\mathrm{p}}e^{\lambda t}$ (24)

$\mathrm{I(t)} = \mathrm{z} + \mathrm{q(t)}; \mathrm{q(t)} = \bar{\mathrm{q}}e^{\lambda t}$ (25)

$\mathrm{R(t)} = \mathrm{y} + \mathrm{r(t)}; \mathrm{r(t)} = \bar{\mathrm{r}}e^{\lambda t}$ (26)

Where $\bar{p}, \bar{q}, \bar{r}$ are constants

$$\ell(t,a) = \phi(a) + \eta(a)e^{\lambda t} \qquad (27)$$

With

$$\bar{q} = \int_0^T \eta(a)da \qquad (28)$$

Substituting (24) to (28) into the model equations (1) to (3)

$$\frac{d}{dt}[x+\bar{p}e^{\lambda t}] = \beta[x+\bar{p}e^{\lambda t}] + \beta[y+\bar{r}e^{\lambda t}] - \gamma(x+\bar{p}e^{\lambda t}) - \mu(x+\bar{p}e^{\lambda t}) + \theta\beta(z+\bar{q}e^{\lambda t}) - \alpha[\alpha+\bar{p}e^{\lambda t}][z] \quad (29)$$

Taking cognizance of equation (10) and neglecting term of order 2 gives

$$\lambda\bar{p}e^{\lambda t} = \beta\bar{p}e^{\lambda t} + \beta\bar{r}e^{\lambda t} - \gamma\bar{p}e^{\lambda t} - \mu\bar{p}e^{\lambda t} + \ell\beta\bar{q}e^{\lambda t} - \alpha x\bar{q}e^{\lambda t} - \alpha z\bar{p}e^{\lambda t}$$

$$0 = (\beta-\gamma-\mu-\lambda-\alpha z)\bar{p} + \beta r + (0\beta - \alpha x)\bar{q} \qquad (30)$$

Also from equation (2) and taking cognizance of equation (11)

$$\gamma\bar{p} - (\lambda+\mu)\bar{r} = 0 \quad (31)$$

Also equation (27) and (28) when substituted into (3) gives

$$\frac{\partial}{\partial t}[\phi(a)+\eta(a)e^{\lambda t}] + \frac{\partial}{\partial a}[\phi(a)+\eta(a)e^{\lambda t}] + h(a)[\phi(a)+\eta(a)e^{\lambda t}] = 0$$

Using equation (12), this reduces to

$$\lambda\eta(a)e^{\lambda t} + e^{\lambda t}\frac{d\eta(a)}{da} + h(a)\eta(a)e^{\lambda t} = 0$$

$$\frac{d\eta(a)}{da} + (h(a)+\lambda)\eta(a) = 0 \quad (32)$$

Solving the ordinary differential equation (32) gives

$$\eta(a) = \eta(0)\exp\{-\int_0^a(\lambda+h(s))ds \quad (33)$$

Integrating (33) over [0,T] gives

$$\int_0^T \eta(a)da = \eta(0)\int_0^T exp\{-\int_0^a (\lambda+h(s))ds\}da$$

Using equation (28), this reduces to

$$q = \int_0^T \eta(a)da = \eta(0)\int_0^T exp\{-\int_0^T \lambda + h(s)ds\}da$$

$$q = \eta(0)b(\lambda) \quad (34)$$

With
$b(\lambda) = \int_0^T [\exp\{-\int_0^a (\lambda + h(s))ds\}]da$ (35)

We now find η(0) as follow

From equation (9)

$$\phi(0) = \alpha xz + (1-\theta)\beta z$$

From equation (27)

$$\ell(t,0) = \phi(a) + \eta(a)e^{\lambda t}$$

$$\ell(t,a) = \phi(0) + \eta(a)e^{\lambda t} = B(t) \quad (36)$$

And $B(t) = \alpha s(t)I(t) + (1-\theta)\beta I(t)$

Substitute (24) to (28) into (6) and use (9) and (36) to get

$$B(t) = \alpha xz + \alpha x\bar{q}e^{\lambda t} + \alpha\bar{p}e^{\lambda t} + \alpha z\bar{p}\bar{q}e^{2\lambda t} + (1-\theta)\beta z + (1-\theta)\beta\bar{q}e^{\lambda t}$$

Compare this with (36) using (9) for ∅(0) gives

$$\emptyset(0) + n(0)e^{\lambda t} = B(t)$$

$$\alpha xz + (1-\theta)\beta z + \eta(0)e^{\lambda t} = \alpha xz + \alpha x\bar{q}e^{\lambda t} + \alpha z\bar{p}e^{\lambda t} + (1-\theta)\beta z + (1-\theta)\beta\bar{q}e^{\lambda t}$$

$$\eta(0)e^{\lambda t} = \alpha x\bar{q}e^{\lambda t} + \alpha z\bar{p}e^{\lambda t} + \alpha\bar{p}\bar{q}e^{2\lambda t} + (1-\theta)\beta\bar{q}e^{\lambda t}$$

Neglecting terms of order 2 gives

$$\eta(0) = \alpha x\bar{q} + \alpha z\bar{p} + (1-\theta)\beta\bar{q} \quad (37)$$

Substituting for $n(0)$ *in* (32) *gives*

$$\bar{q} = [\alpha x\bar{q} + z\bar{p} + (1-\theta)\beta\bar{q}]\, b(\lambda) \quad (38)$$

$$\alpha zb(\lambda)\bar{p} + [(\alpha x + (1-\theta)\beta)b(\lambda) - 1]\bar{q} = 0 \quad (39)$$

The co-efficients of $\bar{p}, \bar{r}, and\, \bar{q}$ in (30), (31) and (39) give the Jacobian determinant.

$$\begin{vmatrix} \beta-\gamma-\mu-\lambda-\alpha z & \beta & \theta\beta-\alpha x \\ \gamma & -(\lambda+\mu) & 0 \\ \alpha zb(\lambda) & 0 & (\alpha x + (1-\theta)\beta)b(\lambda)-1 \end{vmatrix} = 0 \quad (40)$$

and the characteristics equation for the model is therefore given by

$$[[(\lambda+\mu)(\beta-\gamma-\mu-\alpha z) + \gamma\beta)][(\alpha \times +(1-\theta)\beta)(\lambda) - 1] - \alpha z(\lambda+\mu)(\theta\beta-\alpha x)b(\lambda) = 0 \quad (41)$$

5. Stability of the Zero Equilibrium State.

At the zero equilibrium state (x, y, z) = (0, 0, 0), the characteristics equation (41) takes the form;

$$[(\mu+\lambda)(\beta-\mu-\gamma-\lambda) + \beta\gamma][(1-0)\beta b(\lambda) - 1] = 0 \quad (42)$$

With

$$(\mu+\lambda)(\beta-\mu-\gamma-\lambda) + \beta y = 0 \quad (43)$$

$$or\ (1-\theta)\beta b(\lambda) - 1 = 0 \quad (44)$$

Equation (43) is a quadratic equation in λ with negative roots, when

β<μ.

In order to investigate the nature of the root of (44) from equation (35)

$$b(\lambda) = \int_0^T \left[exp\left\{-\int_0^a (\lambda + h(s))ds\right\}\right]da$$

$$= \int_0^T e^{-\lambda a}\,\pi(a)da \quad (45)$$

Applying the result of Bellman and Cooke[3] to analyse the equation (44) for the stability or otherwise of the zero state. Let equation (44) take the form

$$h(\lambda) = (1-\theta)\beta b(\lambda) - 1 = 0 \quad (46)$$

if we set $\lambda = i\omega$ *in* (46) *we have that*

$$h(i\omega) = h_1(\omega) + ih_2(\omega) \quad (47)$$

the condition for Reλ<0 will then be given by the inequality.

$$h_1(0)h_2'(0) - h_1'(0)h_2(0) > 0 \qquad (48)$$

Form (45)

$$b(i\omega) = \int_0^T exp\left\{-\int_0^a \left(i\omega + h(s)\right)ds\right\}da$$

$$= \int_0^T e^{-i\omega a}\,\pi(a)da$$

$$= \int_0^T [(Cos\omega a - isin\omega a)]\pi(a)da$$

$$= f(\omega) + ig(\omega) \qquad (49)$$

And so

$$f(\omega) = \int_0^{T\bar{\pi}(a)} \pi cos\omega ada \qquad (50)$$

$$g(\omega) = -\int_0^T \pi(a)sin\omega ada \qquad (51)$$

And so

$$f(0) = \int_0^T \pi(a)da = \pi;\; g(0) = 0 \quad (52)$$

Also

$$f'(\omega) = \int_0^T a\pi(a)sin\omega ada \quad (53)$$

$$and\ so\ f'(0) = 0 \quad (54)$$

$$g'(\omega) = -\int_0^T a\pi(a)cos\omega ada \qquad (55)$$

And

$$g'(0) = -\int_0^T a\pi\,(a)da = -A \quad (56)$$

From the above equations:

$$h_1(\omega) = (1-\theta)\beta f(\omega) - 1 \qquad (57)$$

$$h_2(\omega) = (1-\theta)\beta\, g(\omega) \qquad (58)$$

$$h_1'(\omega) = (1-\theta)\beta f'(w) \qquad (59)$$

$$h_2'(\omega) = (1-\theta)\beta g'(\omega) \qquad (60)$$

$$h_1(0) = (1-\theta)\beta\pi - 1 \qquad (61)$$

$$h_2(0) = 0 \qquad (62)$$

$$h_1'(0) = 0 \qquad (63)$$

$$h_2'(0) = -(1-\theta)\beta A \qquad (64)$$

The inequality (48) gives

$$[(1-\theta)\beta\pi - 1][(1-\theta)\beta A] < 0 \qquad (65)$$

$$since\ (1-\theta)\beta A > 0 \qquad (66)$$

The inequality (65) holds if

$$(1-\theta)\beta\bar{\pi} - 1 < 0 \qquad (67)$$

We define after simplifying and substituting, that

$$J_{\prime}(k) = (1-\theta)\beta\left(\mu + \delta\tan\frac{\pi}{2k}\right) - \mu\left(\mu + \delta\tan\frac{\pi}{2k}\right) - (1 - \theta)\beta\mu\exp\left(\frac{2\delta TK}{\pi}\ln\left|\cos\frac{\pi}{2k}\right| - \mu T\right) \quad (68)$$

$$so\ the\ origin\ will\ be\ stable\ when\ \beta < \mu\ and\ J_{\prime}(K) < 0 \quad (69)$$

6. Stability of The Non-Zero Equilibrium State.

For the stability or non stability of the non-zero state, let the characteristics equation (41) takes the form

$$H(\lambda) = 0 \qquad (70)$$

If we set $\lambda = i\omega\ in\ (70) we\ have\ that$

$$H(i\omega) = F(\omega) + iG(\omega) \qquad (71)$$

And from (35)

$$b(i\omega) = \int_0^T exp\left\{-\int_0^a (i\omega + h(s))\,ds\right\}da$$

$$= \int_0^T e^{i\omega a}\pi(a)da \qquad (72)$$

$$= \int_0^T (cos\omega a - isin\omega a)\pi(a)da$$

$$= f(\omega) + ig(\omega)$$

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And so

$$f(\omega) = \int_0^T \pi\,(a) cos\omega a da \qquad (73)$$

$$g(\omega) = -\int_0^T \pi(a) sin\omega a da \qquad (74)$$

And so

$$f(0) = \int_0^T \pi(a)da = \bar{\pi}; \qquad g(0) = 0 \quad (75)$$

$$f^{'}(\omega) = \int_0^T a\pi(a) sin\omega\, ada \qquad (76)$$

$$f^{'}(0) = 0 \qquad (77)$$

$$g^{'(\omega)} = \int_0^T a\pi(a) cos\omega a da \qquad (78)$$

And

$$g^{'}(0) = -\int_0^T a\pi\,(a)da = -A \qquad (79)$$

$H(i\omega)$ is thus given by:

$$H(i\omega) = [(\mu + i\omega)(\beta - \mu - \gamma - \alpha z - i\omega) + \beta\gamma][(\alpha x + (1-\theta)\beta)(f(\omega) + ig(\omega) - 1)]$$

$$-\alpha z(\mu + i\omega)(\theta\beta - \alpha x)\big(f(\omega) + ig(\omega)\big) \qquad (80)$$

$$F(\omega) = \mu(\beta - \mu - \gamma - \alpha z) f(\omega)(\alpha x + (1-\theta)\beta) - \mu(\beta - \mu - \gamma - \alpha z) + \omega\, f(w)(\alpha x + (1-\theta)\beta)$$

$$\omega - \omega(\beta - 2\mu - \gamma - \alpha z)(\alpha x + (1-\theta)\beta)g(\omega) + \beta\gamma\big(\alpha x + (1-\theta)\beta\big)f(\omega) - \beta\gamma - \alpha x(\theta\beta - \alpha x)(\mu f(\omega) - \omega g(\omega)\big) \qquad (81)$$

$$G(\omega) = \mu(\beta - \mu - \gamma - \alpha z)(\alpha x + (1-\theta)\beta)g(\omega) + \omega\,(\alpha x + (1-\theta)\beta)g(\omega) + \omega(\beta - 2\mu - \gamma\alpha z)$$

$$(\alpha x + (1-\theta)\beta)f(\omega) - \omega(\beta - 2\mu - \gamma - \alpha z) + \beta\gamma(\alpha x(1-\theta)\beta)g(\omega) - \alpha z(\theta\beta - \alpha x)\big(\mu g(\omega) + \omega f(\omega)\big) \qquad (82)$$

And so

$$G(0) = 0$$

And

$$F(0) = \mu(\beta - \mu - \gamma - \alpha z)(\alpha x + (1-\theta)\beta\pi - \mu(\beta - \mu - \gamma - \alpha z) + \beta\gamma(\alpha X + (1-\theta)\beta)\pi - \beta\gamma - \alpha z(\theta\beta - \alpha x)\beta \qquad (83)$$

From $G^{'}(\omega)$, we obtain

$$G^{'}(0) = -\mu(\beta - \mu - \gamma - \alpha z)(\alpha x + (1-\theta)\beta)A - \beta\gamma(\alpha X + (1-\theta)\beta)A - (\beta - 2\mu - \gamma - \alpha z) + \alpha\mu z(\theta\beta - \alpha x)A \quad (84)$$

We now need to obtain the condition for which the inequality.

$$F(0)G^{'}(0) - F^{'}(0)G(0) > 0 \qquad (85)$$

Holds i.e

$$F(0)G^{'}(0) > 0 \qquad (86)$$

Since G(0) = 0. The expression F(0) $G\,(0\,)$ is cumbersome and therefore the nature of the sign is not clearly decipherable, hence we shall examine this condition in the neighbourhood of $\alpha = 0.$ *so $F(0)$ and $G^{'}(0)$ take the forms:*

$$F(0) = (\beta - \mu)(\mu + \gamma)[(1-\theta)\beta\pi - 1] = (\beta - \mu)(\mu + \gamma)J(k) \quad (87)$$

So

$$Sgn\, F(0) = SgnJ(k) \qquad (88)$$

$$Also\ G^{'}(0) = (\beta - \mu)(\mu + \gamma)(1-\theta)\beta A - (\beta - 2\mu - \gamma) \quad (89)$$

Taking

$(\beta - 2\mu - \gamma) < 0$

which is informed by the dominance of the parameters 2μ and γ,

$$G^{'}(0) > 0$$

if $2\mu + \gamma - \beta > (\beta - \mu)(\mu + \gamma)(1-\theta)\beta A$ (90)

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Hence if J(K) <0, the non-zero state will be unstable, and if otherwise it will be stable at least locally.

Conclusion

We noted that for J(K)<0, K must be very low, this gives the condition for the stability of the origin and the instability of the non-zero state. On the other hand when J(K)>0, K must be correspondingly high, which leads to the instability of the origin and the possible stability of the non-zero state, locally.

A low level of K indicates high rate of death among the infected, hence the stability of the origin leading to a wiping out of the population. While a high level of K indicates longer life span for the infected and hence the instability of the origin and stability of the non zero state. This suggests a usefulness of the use of measures like the anti retroviral drugs to enhance the life-span of the infected.

From the inequality (90), we obtain a bound for γ, the measure of public campaign effectiveness, given by:

$$\gamma > \frac{(\mu\beta - \mu)(1-\theta)\beta A + \beta - 2\mu}{(1-\beta-\mu)(1-\theta)\beta A} \qquad (91)$$

And the inequalities

$$J(K_c) > 0 \qquad (92)$$

and

$$\gamma_c > \frac{(\mu\beta - \mu)(1-0)\beta A + \beta - 2\mu}{(1-\beta-\mu)(1-\theta)\beta A} \qquad (93)$$

give the critical values for K and γ respectively. This suggests that the public awareness campaign should target a success level not below the value J(k_c) while the slowing down of infected victims death should not slide below γ_c. This may ensure the non-extinction of the population.

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