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A Thermo-Acoustic Model of a Gas Turbine Combustor Driven by a Loud-Speaker

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***Abstract*—We develop a thermo-acoustic model of a gas turbine combustor where acoustic perturbations are generated by an external loud speaker. Our analysis shows how the electro-mechanical dynamics of a speaker effects on the overall thermo-acoustic dynamics of a combustor. A numerical example is included to illustrate our results.**

***Keywords*—One-dimensional acoustic model, Acoustic Transfer Matrix**

I. Introduction

Gas turbine combustor are prone to combustion instability where the dynamics of combustion and that of thermo-acoustic field have a positive feedback (synergy effect) with each other to result in a large magnitude of pressure and velocity perturbations [1].

For a theoretical investigation of a possible occurrence of combustion instability it is essential to develop precise dynamic models of both combustor thermo-acoustic and combustion process. The thermo-acoustic part is commonly approximated with one-dimensional acoustics and this approximation allows an analytical model with relatively ease.

In contrast, however, the dynamic properties of a combustion process is so complicated and highly nonlinear that in many cases experimental approach are preferred. For an experimental identification of the combustion dynamics, a loud speaker is widely employed to generated a user-controllable acoustic perturbations and thus to obtain a frequency response of combustion flame in terms of heat rate perturbation with respect to a velocity perturbation, i.e., the so-called flame transfer function.

The use of a loud speaker however effects the thermo-acoustic property of a combustor system without a speaker since the speaker serves as a new boundary condition, which leaves us an important question ; *how the speaker dynamics will modify the thermo-acoustics of combustor ?*. This is a key motivation of our development presented in this paper.

Our thermo-acoustic model for a combustor also allows us to indirectly and experimentally validate the soundness of our acoustic model in the process of obtaining a flame transfer function.

II. Acoustic Model

A. Speaker Model

A speaker can be seen as a RL (resistor-inductance) circuit whose dynamics can be described as

$$V = Ri + L\frac{di}{dt} + k_b\frac{d\xi}{dt} \tag{1}$$

where t denotes time, R and L are electrical resistance and inductance, $i(t)$ is current, $V(t)$ is a driving voltage, k_b is the coefficient of the counter-electromotive force and ξ denotes the position of speaker diaphragm (cone) which is assumed to have a simple one-dimensional motion.

The mechanical force generated by the speaker coil is $f = k_f i$ for a constant k_f and it drives a mechanical vibration system

$$\begin{aligned} f = k_f i &= m\ddot{\xi} + b\dot{\xi} + k\xi + A_e p'(x,t)|_{x=0} \\ \dot{\xi}(t) &= u'(x,t)|_{x=0} \end{aligned} \tag{2}$$

where x denotes the one-dimensional coordinate of an acoustic element attached to the speaker such that $x = 0$ correspond to $\xi = 0$, A_e denotes the effective area of the speaker cone, constants m, b, k are mechanical parameters of the speaker. In addition $p'(x,t)$ and $u'(x,t)$

are the perturbations of pressure and velocity. See an illustration in Fig. 1.

Using Laplace transformation we rewrite (1)-(2) as

$$V(s) = (Ls + R)I(s) + k_b u'(0,t)$$
$$k_f I(s) = \left(ms + \frac{k}{s} + b\right) u'(0,s) + A_e p'(0,s) \tag{3}$$

and an elimination of the term $I(s)$ in those two equalities gives

$$V(s) = \begin{bmatrix} 1/G_p & 1/G_u \end{bmatrix} \begin{bmatrix} p'(0,s) \\ \nu'(0,s) \end{bmatrix}, \quad \nu := \overline{\rho} c u' \tag{4}$$

$$G_p(s) := \frac{k_f}{(Ls+R)A_e}$$
$$G_u(s) := \frac{\overline{\rho} c k_f s}{k_b k_f s + (Ls+R)(ms^2+bs+k)} \tag{5}$$

and $\overline{\rho}$, $\overline{u}$ denote the mean density, velocity and c denotes the sound speed.

B. A Simple Duct Model

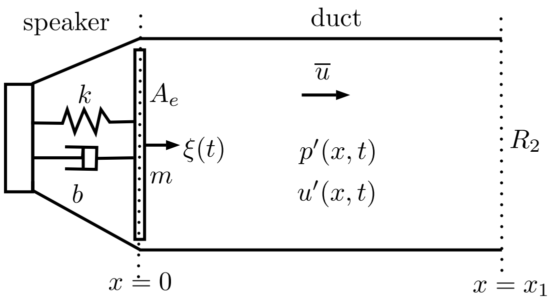

Fig. 1: Duct driven by a speaker

The wave propagation over the interval $[0, x_1]$ is given

$$\begin{bmatrix} p'(x_1,t) \\ \nu'(x_1,t) \end{bmatrix} = P_0^{x_1} \begin{bmatrix} p'(0,t) \\ \nu'(0,t) \end{bmatrix} \tag{6}$$

$$P_0^{x_1} = \frac{1}{2} \begin{bmatrix} e^{-\tau^+ s} + e^{\tau^- s} & e^{-\tau^+ s} - e^{\tau^- s} \\ e^{-\tau^+ s} - e^{\tau^- s} & e^{-\tau^+ s} + e^{\tau^- s} \end{bmatrix}, \tag{7}$$

$\tau^{\pm} := x_1/(c \pm \overline{u})$ where c denotes the sound speed.

The boundary condition at x_1 specified with a reflection coefficient R, the ratio of incident and reflected waves at $x = x_1$, can be written as

$$\begin{bmatrix} -1 + R_2 & 1 + R_2 \end{bmatrix} \begin{bmatrix} p'(x_1,t) \\ \nu'(x_1,t) \end{bmatrix} = 0. \tag{8}$$

In addition, a combination of (4) and (6) gives

$$\begin{bmatrix} G_u & G_p \end{bmatrix} (P_0^{x_1})^{-1} \begin{bmatrix} p'(x_1,t) \\ \nu'(x_1,t) \end{bmatrix} = G_u G_p V. \tag{9}$$

From this equation and (8), one can find the following relation between $\nu'(s)$ and $V(s)$;

$$\frac{p'(x_1,s)}{V(s)} = -\frac{1}{2} \cdot \frac{1 + H(s)e^{-\delta s}}{1 - H(s)R_2 e^{-\delta s}} G_u(s) G_p(s) \tag{10}$$

$$\frac{\nu'(x_1,s)}{V(s)} = \frac{1}{2} \cdot \frac{1 - H(s)e^{-\delta s}}{1 - H(s)R_2 e^{-\delta s}} G_u(s) G_p(s) \tag{11}$$

where $\delta = \tau^+ + \tau^-$ and $H(s)$ is given in (15). The transfer function (16) will be used later.

Note that the pole of the above transfer function are composed of two parts ; poles of the electro-mechanical transfer functions $G_u G_p$ in (5) and *acoustic* poles which are the roots of the next equation

$$1 - H(s)R_2 e^{-\delta s} = 0. \tag{12}$$

Let us suppose the speaker is removed and the left reflection coefficient of the duct at $x = 0$ is R_1. In this case it is easy to show that the corresponding acoustic poles are given as roots of

$$1 - R_1 R_2 e^{-\delta s} = 0. \tag{13}$$

A comparison of this fact and (12) suggests that the duct with a speaker in Fig. 1 can be seen as a simple duct along with a *frequency-dependent* reflection coefficient $R_1 = H(s)$.

Note that if

$$b \gg \overline{\rho} c A_e, \tag{14}$$

then we have $H(s) \approx 1$.

$$H(s) := \frac{Lms^3 + [(b - \overline{\rho}c\,A_e)\,L + Rm]\,s^2 + [Lk + (b - \overline{\rho}c\,A_e)\,R + k_b k_f]\,s + Rk}{Lms^3 + [(b + \overline{\rho}c\,A_e)\,L + Rm]\,s^2 + [Lk + (b + \overline{\rho}c\,A_e)\,R + k_b k_f]\,s + Rk} \tag{15}$$

$$K(s) := \frac{2k_f s}{Lms^3 + [(b + \overline{\rho}cA_e)L + Rm]\,s^2 + [kL + (b + \overline{\rho}cA_e)R + k_b k_f]\,s + Rk} \tag{16}$$

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C. Thermo-acoustic Combustor Model

Making use of the previous speaker model, here we develop a thermo-acoustic model for a combustor as illustrated in Fig. 2.

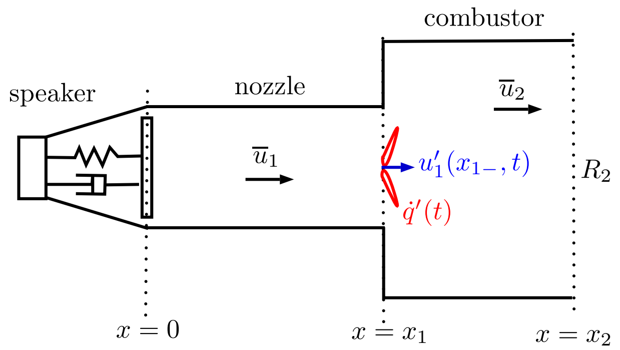

Fig. 2: Acoustic Model of Combustor

Acoustic waves propagate from x_{1+}, just after an area expansion which is the same as the location of a *thin* flame, to the outlet x_3 as

$$\begin{bmatrix} p' \\ \nu' \end{bmatrix}_{x_2} = P_{x_1}^{x_2} \begin{bmatrix} p' \\ \nu' \end{bmatrix}_{x_{1+}} \tag{17}$$

$$P_{x_1}^{x_2} := \frac{1}{2} \begin{bmatrix} e^{-\tau_2^+} + e^{\tau_2^-} & e^{-\tau_2^+} - e^{\tau_2^-} \\ e^{-\tau_2^+} - e^{\tau_2^-} & e^{-\tau_2^+} + e^{\tau_2^-} \end{bmatrix}, \tag{18}$$

$$\tau_2^{\pm} := (x_2 - x_1)/(c_2 \pm \overline{u}_2) \tag{19}$$

where c_2 denotes the sound speed after a flame.

In addition, the right boundary condition at x_2 can be written as

$$\begin{bmatrix} -1 + R_2 & 1 + R_2 \end{bmatrix} \begin{bmatrix} p' \\ \nu' \end{bmatrix}_{x_2} = 0 \tag{20}$$

A combination of (20) and (17) gives

$$\begin{bmatrix} -1 + R_2 & 1 + R_2 \end{bmatrix} P_{x_1}^{x_2} \begin{bmatrix} p' \\ \nu' \end{bmatrix}_{x_{1+}} = 0. \tag{21}$$

By replacing x_1 in (9) with x_{1-}, just before an area expansion *or* flame location, we have

$$\begin{bmatrix} G_u & G_p \end{bmatrix} (P_0^{x_1})^{-1} \begin{bmatrix} p' \\ \nu' \end{bmatrix}_{x_{1-}} = G_u G_p V \tag{22}$$

where $\{c, \overline{u}, \overline{\rho}, \tau, \delta\}$ in the previous section should be rewritten as $\{c_1, \overline{u}_1, \overline{\rho}_1, \tau_1, \delta_1\}$.

The mass, energy and momentum relations across an area expansion and a flame give the following relation

$$\begin{bmatrix} p' \\ \nu' \end{bmatrix}_{x_1^+} = \begin{bmatrix} 1 & \xi_1 M_1 \\ \frac{(1-\alpha^2)\gamma}{\alpha\beta} M_1 & \frac{1}{\alpha\beta} \end{bmatrix} \begin{bmatrix} p' \\ \nu' \end{bmatrix}_{x_1^-} + \begin{bmatrix} \xi_2 M_1 \\ 1 \end{bmatrix} \left(\frac{\gamma - 1}{\alpha\beta c_1} \right) \dot{q}'(t) \tag{23}$$

where $\dot{q}(t)$ denotes the combustion heat source in the unit of [Joule/m^2] and two constant ξ_1, ξ_2 depend on particular acoustic model of the area expansion at x_1 [2], [3]. In this paper, following [2], we have

$$\xi_1 := \frac{2(\beta - 1) + 1 - \alpha^2}{\beta^2}, \quad \xi_2 := -\frac{\alpha}{\beta} \tag{24}$$

where α, β denotes the ratio of sound speed and area before and after the area expansion or flame.

By multiplying the matrix in the left hand side of (22) to (23), one can obtain

$$0 = \begin{bmatrix} -1 + R_2 & 1 + R_2 \end{bmatrix} P_{x_1}^{x_2} \begin{bmatrix} 1 & \xi_1 M_1 \\ \frac{(1-\alpha^2)\gamma}{\alpha\beta} M_1 & \frac{1}{\alpha\beta} \end{bmatrix} \begin{bmatrix} p' \\ \nu' \end{bmatrix}_{x_1^-} + \begin{bmatrix} -1 + R_2 & 1 + R_2 \end{bmatrix} P_{x_1}^{x_2} \begin{bmatrix} \xi_2 M_1 \frac{\gamma-1}{\alpha\beta c_1} \\ \frac{\gamma-1}{\alpha\beta c_1} \end{bmatrix} \dot{q}'(t). \tag{25}$$

This equation and (22) constitute two simultaneous equations between the waves $(p', \nu')_{x_{1-}}$ and two sources $V(s)$ and $\dot{q}'(s)$.

In particular, regarding $\{V, \dot{q}'\}$ as two driving inputs and $u'(x_{1-}) = \nu(x_{1-})/\overline{\rho}_1 c$ as a velocity output, the corresponding MISO (multi-input single-output) transfer function representations can be written as

$$u'(x_{1-}, s) = G_s(s)V(s) + G_f(s)\dot{q}(s) \tag{26}$$

where the *speaker* transfer function $G_s(s)$ and the *heat* transfer function $G_q(s)$ are given

$$G_s(s) := \frac{N_s(s)}{D(s)}, \quad G_q(s) := \frac{N_q(s)}{D(s)} \tag{27}$$

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where

$$N_s(s) := K(s)\psi_1(1-\psi_2 R_2 e^{-\delta_2 s})e^{-\delta_1 s} \quad (28)$$

$$N_f(s) := -\frac{\gamma-1}{\overline{\rho}c^2}\psi_3(1-H(s)e^{-\delta_1 s})(1+\psi_4 R_2 e^{-\delta_2 s}) \quad (29)$$

$$D(s) := 1-\psi_6 R_2 e^{-\delta_2 s} + \psi_7 H(s)(1-\psi_5 R_2 e^{-\delta_2 s})e^{-\delta_1 s}, \quad (30)$$

$\delta_i := \tau_i^+ + \tau_i^-$ $(i=1,2)$, $K(s)$ in (16) and

$$\begin{aligned}
\psi_1 &:= \frac{\alpha\beta+(\alpha^2-1)\gamma M_1}{\alpha\beta+1+(\alpha^2\gamma-\alpha\beta\xi_1-\gamma)M_1}\\
\psi_2 &:= \frac{\alpha\beta-(\alpha^2-1)\gamma M_1}{\alpha\beta+(\alpha^2-1)\gamma M_1}\\
\psi_3 &:= \frac{1-\xi_2 M_1}{\alpha\beta+1+(\alpha^2\gamma-\alpha\beta\xi_1-\gamma)M_1}\\
\psi_4 &:= \frac{1+\xi_2 M_1}{1-\xi_2 M_1}\\
\psi_5 &:= \frac{\alpha\beta+1-(\alpha^2\gamma-\alpha\beta\xi_1-\gamma)M_1}{\alpha\beta-1+(\alpha^2\gamma+\alpha\beta\xi_1-\gamma)M_1}\\
\psi_6 &:= \frac{\alpha\beta-1-(\alpha^2\gamma+\alpha\beta\xi_1-\gamma)M_1}{\alpha\beta+1+(\alpha^2\gamma-\alpha\beta\xi_1-\gamma)M_1}\\
\psi_7 &:= \frac{\alpha\beta-1+(\alpha^2\gamma+\alpha\beta\xi_1-\gamma)M_1}{\alpha\beta+1+(\alpha^2\gamma-\alpha\beta\xi_1-\gamma)M_1}
\end{aligned} \quad (31)$$

It is a common situation where $\alpha\beta \gg 1$ and $M_1 \approx 0$. In this case we have $\psi_i = 0$ for all $i = 1, \cdots, 7$ except $\psi_3 \approx 1/\alpha\beta$ and thus obtain simple representations;

$$\begin{aligned}
N_s(s) &= K(s)(1-R_2 e^{-\delta_2 s})e^{-\delta_1 s},\\
N_f(s) &= -\frac{\gamma-1}{\overline{\rho}c^2}(1-H(s)e^{-\delta_1 s})(1+R_2 e^{-\delta_2 s}),\\
D(s) &= (1+H(s)e^{-\delta_1 s})(1-R_2 e^{-\delta_2 s}).
\end{aligned} \quad (32)$$

D. A Numerical Example

We computed various transfer function appeared in our developments with speaker and combustor parameters summarised in Table I.

Two Bode plots of the transfer function $H(s)$ and $K(s)$ shown in Fig. 3-4 shows minimal and maximal peaks, respectively, which seems to come from the mechanical speaker resonance frequency

$$f_s := \frac{1}{2\pi}\sqrt{\frac{k}{m}} = 35.59 \text{ (Hz)}. \quad (33)$$

$\gamma = 1.38$	$\alpha = 6$	$\overline{u}_1 = 60$	$c_1 = 345$
$\overline{\rho}_1 = 1.177$	$\overline{\rho}_2 = 0.0327$	$L_1 = 0.6$	$L_2 = 1.5$
$L = 0.02$	$R = 4$	$m = 0.01$	$b = 0.6$
$k = 500$	$A_e = A_1$	$k_b = 1$	$k_f = 0.5$
$\beta = 10$	$A_1 = 0.005$	$A_2 = 0.05$	$R_2 = -1$

TABLE I: Parameters for the example

This resonance frequency clearly effects on the Bolde plots of the speaker transfer function $G_s(s)$ in Fig. 6 and $G_f(s)$ in Fig. 5.

The second peak near 157 (Hz) in Fig. 6 and Fig. 5 seems to be an acoustic resonance frequency of the nozzle part $(0, x_1)$. In fact, the sudden area expansion and the flame at x_1 result in an acoustic impedance jump and thus acoustic waves in either nozzle or combustor are largely reflected at x_1 [4] (p. 151). This gives rise to a standing wave inside the nozzle whose resonance frequency is given

$$f_q := \frac{1+2k}{2\delta_1}, \quad k = 0, 1, \cdots \quad (34)$$

and the smallest frequency $f_r \approx 140$ (Hz) with $k = 0$ is close to the observed peak frequency.

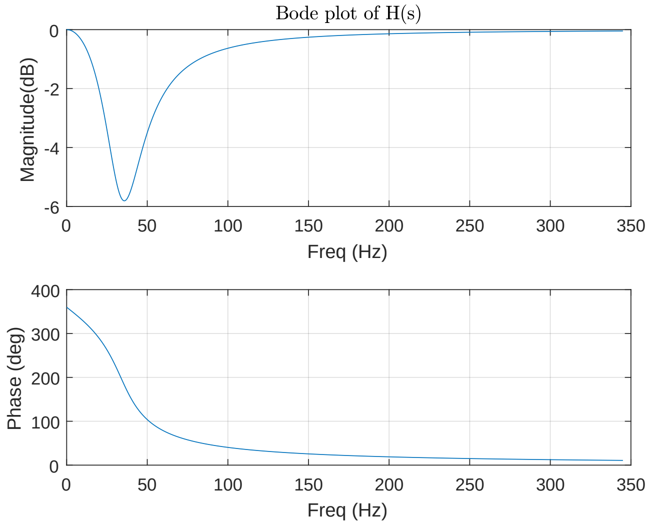

Fig. 3: Bode plot of $H(s)$

III. Conclusion

We developed a thermo-acoustic model of a gas turbine combustor under the condition that the nozzle inlet

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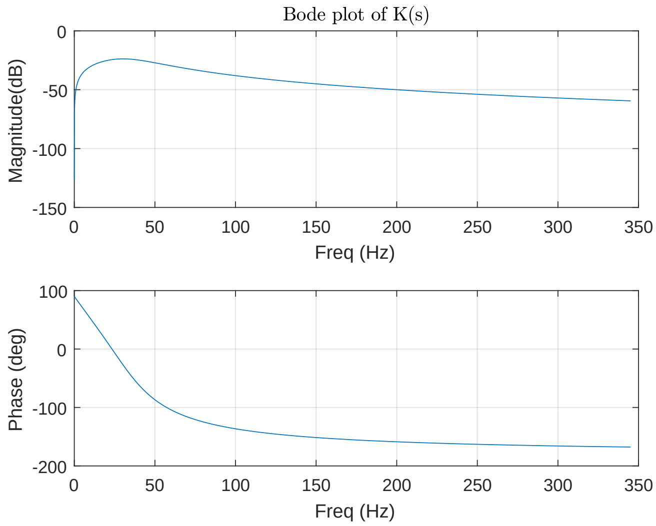

Fig. 4: Bode plot of $K(s)$

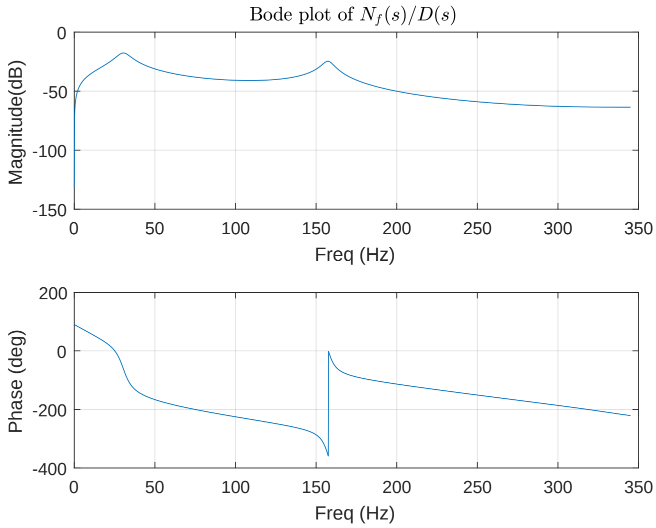

Fig. 5: Bode plot of $G_s(s)$

is acoustically driven by a loud speaker. Our model was given as a transfer function representation in which an electrical voltage for speaker and a heat perturbation are two inputs and the velocity perturbation at the flame location is a single output. A numerical case study suggested a possibility that the thermo-acoustic properties of the combustor can be effected by the resonance dynamics of the speaker. It is left as a further work to validate our model with real-world combustor systems.

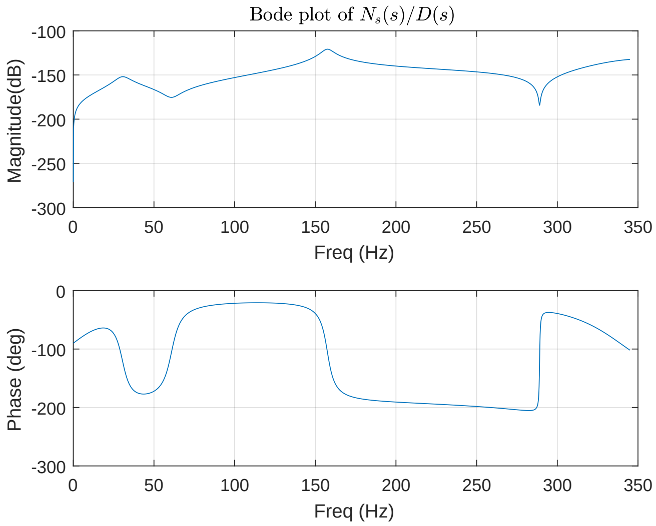

Fig. 6: Bode plot of $G_q(s)$

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