A Study On C*G-Closed Sets In Bitopological Spaces

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Abstract

In this paper, we have introduced the concept of c*g –closed and some of their properties in bitopological space.

Key words: (I,j)-c*g-closed
1. INTRODUCTION

A triple \((x, \tau_1, \tau_2)\) where \(X\) is non-empty set and \(\tau_1\) and \(\tau_2\) are topologies on \(X\) is called a bitopological space and Kelly \([11]\) initiated the study of such spaces. In 1985 Fukutake \([5]\) introduced the concepts of \(g\)-closed sets in bitopological spaces and after that several authors turned their attention to the generalization of various concepts of topology by considering bitopological spaces instead of topological spaces. In 2004, P.Sundaram \([12]\) introduced the concept of \(g^*-\)closed sets in bitopological spaces.

Throughout this chapter \((X, \tau_1, \tau_2)\) (or \(X\))and \((Y, \sigma_1, \sigma_2)\) (or \(Y\)) denote two non empty bitopological spaces. In this section we introduce the concept of \((i,j)-c^*g\)-closed sets and we obtain some interesting results in bitopological spaces.

2. PRELIMINARIES

DEFINITION 2.1: A subset \(A\) of \(X\) is called

i) \((i,j)^*-\)generalized closed (briefly \((i,j)^*-\)g-closed) \([5]\) if \(\tau_j - \text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(\tau_i\)-open in \(X\).

ii) \((i,j)-\) regular generalized closed (briefly \((i,j)-\)rg-closed) \([1]\) if \(\tau_j - \text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(\tau_i\)-regular open in \(X\).

iii) \((i,j)-\)generalized pre regular closed (briefly \((i,j)-\)gpr-closed) \([7]\) if \(\text{pcl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(\tau_i\)-regular open in \(X\).

iv) \((i,j)-\)weakly generalized closed (briefly \((i,j)-\)wg-closed) \([6]\) if \(\tau_j - \text{cl}(\text{int}(A)) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(\tau_i\)-open in \(X\).

v) \((i,j)-\)strongly generalized closed (briefly \((i,j)-\)strongly g-closed) \([12]\) if \(\tau_j - \text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(\tau_i\)-g-open in \(X\).

vi) \((i,j)-\)weakly closed (briefly \((i,j)-\)w-closed) \([7]\) if \(\tau_j - \text{cl}(A) \subseteq U\)
whenever \( A \subseteq U \) and U is \( \tau_i \)-semi open in X.

vii) \((i,j)\)-generalized \( \alpha \)-closed (briefly \((i,j)\)-\(g\alpha\)-closed) [3] if \( \tau_j\text{-}\text{acl}(A) \subseteq U \) whenever \( A \subseteq U \) and U is \( \tau_i\)-\( \alpha \)-open in X.

viii) \((i,j)\) generalized semi-closed (briefly \((i,j)\)-\(gs\)-closed) [3] if \( \tau_j\text{-}\text{scl}(A) \subseteq U \) whenever \( A \subseteq U \) and U is \( \tau_i\)-open in X.

3. \(c^*g\)-CLOSED -CLOSED SETS IN BITOPOLOGICAL SPACES

DEFINITION 3.1: A subset A of a bitopological space \((X, \tau_i, \tau_j)\) is said to be an \((i,j)\)-\(c^*g\)-closed set if \( \tau_j\text{-}\text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and U is \( \tau_i\)-\(c^*\)-set in X.

We denote the family of all \((i,j)\)-\(c^*g\)-closed sets in \((X, \tau_i, \tau_j)\) by \(C^*(i,j)\).

THEOREM 3.2: Every \( \tau_j\)-closed set in \((X, \tau_i, \tau_j)\) is \((i,j)\)-\(c^*g\)-closed set in \((X, \tau_i, \tau_j)\) but not conversely.

Proof: - Let A be a \( \tau_j\)-closed set in X. Let U be a \( \tau_i\)-\(c^*\)-set such that \( A \subseteq U \). Since A is \( \tau_j\)-closed, that is \( \tau_j\text{-}\text{cl}(A) = A \). Therefore \( \tau_j\text{-}\text{cl}(A) \subseteq U \). Hence A is \((i,j)\)-\(c^*g\)-closed set in X.

The converse of the above theorem need not be true as seen from the following example.

EXAMPLE 3.3: Consider the topological space \( X = \{a,b,c\} \) with the topologies \( \tau_i = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\} \); \( \tau_j = \{\emptyset, X, \{c\}, \{a, b\}\} \). The set \( \{b\} \) is \((1,2)\)-\(c^*g\)-closed set but not \( \tau_2\)-closed.

THEOREM 3.4: Union of two \((i,j)\)-\(c^*g\)-closed sets is \((i,j)\)-\(c^*g\)-closed.

Proof: - Let A and B be \((i,j)\)-\(c^*g\)-closed sets in X. Let U be a \( \tau_i\)-\(c^*\)-set in X. such that \( A \cup B \subseteq U \). Then \( A \subseteq U \) and \( B \subseteq U \). Since A and B are \((i,j)\)-\(c^*g\)-closed, \( \tau_j\text{-}\text{cl}(A) \subseteq U \) and \( \tau_j\text{-}\text{cl}(B) \subseteq U \). Hence \( \tau_j\text{-}\text{cl}(A \cup B) = \tau_j\text{-}\text{cl}(A) \cup \tau_j\text{-}\text{cl}(B) \subseteq U \). Therefore \( A \cup B \) is \((i,j)\)-\(c^*g\)-closed.

REMARK 3.5: Intersection of two \((i,j)\)-\(c^*g\) -closed sets in X need not be \((i,j)\)-\(c^*g\) -closed sets in X is proved in the following example.
EXAMPLE 3.6: Consider the topological space $X = \{a, b, c\}$ with the topologies $\tau_1 = \{\emptyset, X, \{a\}\}$; $\tau_2 = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$. In this topology consider the set $\{b, c\}$ and $\{a, c\}$. Intersection of these two sets not contained in (1,2)$-c*g$-closed sets. Therefore intersection of two (1,2)$-c*g$-closed sets in $X$ is not (1,2)$-c*g$-closed sets in $X$.

THEOREM 3.7: A subset $A$ of $X$ is (i,j)$-c*g$-closed in $X$ if and only if $j-cl(A)/A$ does not contain any non empty $i-c*$-set in $X$.

Proof:- Suppose that $A$ is a (i,j)$-c*g$-closed set in $X$. We prove the result by contradiction. Let $U$ be $i$-$c*$-set such that $U \subseteq \tau_j-cl(A)/A$ and $U \neq \emptyset$. Then $U \subseteq \tau_j-cl(A) \cap A^c$. Therefore $U \subseteq \tau_j-cl(A)$ and $U \subseteq A^c$ is $i$-$c*$-set and $A$ is (i,j)$-c*g$-closed, $\tau_j-cl(A) \subseteq U^c$. That is $U \subseteq [\tau_j-cl(A)]^c$. Hence $U \subseteq \tau_j-cl(A) \cap [\tau_j-l(A)]^c \neq \emptyset$. That is $U \neq \emptyset$. Which is contradiction. Hence $\tau_j-cl(A)/A$ does not contain any non empty $\tau_j-c*$-set in $X$.

Conversely assume that $\tau_j-cl(A)/A$ contains no non empty $\tau_j-c*$-set. Let $A \subseteq U$, $U$ is $\tau_j-c*$-set. Suppose that $\tau_j-cl(A)$ is not contained in $U$. Then $\tau_j-cl(A) \cap U^c$ is a non empty $\tau_j-c*$-set and contained in $\tau_j-cl(A)/A$, which is contradiction. Therefore $\tau_j-cl(A) \subseteq U$. Hence $A$ is (i,j)$-c*g$-closed.

REMARK 3.8: The converse of the above two theorems is not true as seen from the following example.

EXAMPLE 3.9: Consider $X = \{a, b, c\}$ with the topology $\tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$ and $\tau_2 = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}\}$. Let $A = \{a\}$, then $\tau_2-cl(A)/A = \{\{a, c\}/\{a\}\} = \{c\}$ does not contain any non empty $\tau_i-c*$-set, but $A = \{c\}$ is not (1,2)$-c*g$-closed set in $X$.

THEOREM 3.10: Every (i,j)-strongly $g$-closed set in $X$ is a (i,j)$-c*g$-closed set in $X$ but not conversely.

The converse of the above theorem need not be true as seen from the following example.
EXAMPLE 3.11: Consider the topological space $X=\{a,b,c\}$ with the topologies $\tau_1 = \{\emptyset,X,\{a,b\}\}$; $\tau_2 = \{\emptyset,X,\{a\},\{b\},\{a,b\}\}$. Then the set $A= \{a,b\}$ is $(1,2)$-c*g-closed set but not $(1,2)$-strongly g closed.

THEOREM 3.12: Every $(i,j)$-c*g-closed set in $X$ is a $(i,j)$-gpr-closed set in $X$ but not conversely.

The converse of the above theorem need not be true as seen from the following example.

EXAMPLE 3.13: Consider the topological space $X=\{a,b,c\}$ with the topologies $\tau_1 = \{\emptyset,X,\{c\},\{a,b\}\}$ and $\tau_2 = \{\emptyset,X,\{b\},\{b,c\},\{a,b\}\}$. Then the set $A= \{b,c\}$ is not $(1,2)$-c*g-closed set but $(1,2)$-gpr closed.

REMARKS 3.14: From the above theorem and example we get the following diagram.

\[
\begin{array}{cc}
\text{j-closed} & \\
\downarrow & \\
\text{(i,j)-strongly g-closed} & \\
\downarrow & \\
\text{(i,j)-c*g-closed} & \\
\downarrow & \\
\text{(i,j)-gpr-closed.} & \\
\end{array}
\]

In the above diagram none of the implications can be reversed.

REMARK 3.15:

The concept of $(i,j)$-c*g-set is independent of the following classes of sets namely $(i,j)$-$\alpha$-closed, $(i,j)$-$g$-closed, $(i,j)$-wg-closed, $(i,j)$-$\beta$-closed, $(i,j)$-$g\alpha$-closed, $(i,j)$-$\omega$-closed, $(i,j)$-pre-closed and $(i,j)$-$\alpha g$-closed.

EXAMPLE 3.16: Consider the topological space $X=\{a,b,c\}$ with the topologies $\tau_1 = \{\emptyset,X,\{a\}\}$ and $\tau_2 = \{\emptyset,X,\{a\},\{c\},\{a,c\}\}$. Then the set $A= \{c\}$ is not $(1,2)$-c*g-closed set but $(1,2)$-$\alpha$-closed, $(1,2)$-$\beta$-closed, $(1,2)$-pre-closed and $(1,2)$-semi-closed. In the same topologies the set $A= \{a\}$ is $(1,2)$-c*g-closed set but not $(1,2)$-$\alpha$-closed, not $(1,2)$-$\beta$-closed, not $(1,2)$-pre-closed and not $(1,2)$-semi-closed.

EXAMPLE 3.17: Consider the topological space $X=\{a,b,c\}$ with the topologies $\tau_1 = \{\emptyset,X,\{b\}\}$ and $\tau_2 = \{\emptyset,X,\{b\},\{c\},\{b,c\}\}$. Then the set $A= \{c\}$ is not $(1,2)$-c*g-closed set but
(1,2)- g- closed and the set \{a,b\} is (1,2)-c*g- closed set but not (1,2)- g-
closed.

**EXAMPLE 3.18:** Consider the topological space \(X=\{a,b,c\}\) with the
topologies \(\tau_1 = \{\phi, X, \{a\}\}\) and
\(\tau_2 = \{\phi, X, \{b\}\}\). Then
In this bitopologies the set \(A = \{c\}\) is not (1,2)-c*g- closed set but (1,2)- w-
closed and (1,2)- wg- closed. For the same topologies,
The set \(A = \{a,b\}\) is (1,2)-c*g- closed set but not (1,2)- w-
closed.
The set \(A = \{b,c\}\) is (1,2)-c*g- closed set but not (1,2)- wg- closed.

**EXAMPLE 3.19:** Consider the topological space \(X=\{a,b,c\}\) with the
topologies \(\tau_1 = \{\phi, X, \{b\}, \{c\}, \{b,c\}\}\) and
\(\tau_2 = \{\phi, X, \{b\}\}\). Then
The set \(A = \{c\}\) is not (1,2)-c*g- closed set but (1,2)- gs- closed and (1,2)- sg-
closed.
The set \(A = \{a,b\}\) is (1,2)-c*g- closed set but not (1,2)- gs- closed and (1,2)- sg-
closed.

**REMARK 3.20:** From the above discussion and known results we have
the following diagram.

![Diagram](image)

**REMARK 3.21:**
The concept of (i,j)-c*g \(-\)closed set is independent of the following classes of
sets namely \(\tau_j\)-strongly g- closed, \(\tau_j\)- rg-closed and \(\tau_j\)-gpr \(-\)closed.

**EXAMPLE 3.22:** Consider the topological space \(X=\{a,b,c\}\) with the
topologies \(\tau_1 = \{\phi, X, \{a\}\}\) and
\(\tau_2 = \{\phi, X, \{a\}, \{c\}, \{a,c\}\}\). Then the set
\(A = \{a\}\) is (1,2)-c*g- closed set but not \(\tau_2\)-strongly g- closed, \(\tau_2\)-rg-closed and
\(\tau_2\)-gpr \(-\)closed.
Consider another topologies $\tau_1 = \{ \emptyset, X, \{b\}, \{a,c\} \}$ and $\tau_2 = \{ \emptyset, X, \{a,c\}\}$. Then the set $A = \{a,b\}$ is $\tau_2$-strongly g-closed, $\tau_2$-rg-closed and $\tau_2$-gpr-closed but not $(1,2)$-c*g-closed set.

**REMARK 3.23:** From the above discussion and known results we have the following diagram.

1. $\tau_2$-strongly g-closed
2. $\tau_2$-rg-closed
3. $\tau_2$-gpr-closed
4. $(1,2)$-c*g-closed

**EXAMPLE 3.26:** Consider the topological space $X=\{a,b,c\}$ with the topologies $\tau_1 = \{ \emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}\}$ and $\tau_2 = \{ \emptyset, X, \{a\}\}$. Then the set $A = \{a,c\}$ is $(1,2)$-c*g-closed set but not $\tau_1$-rg-closed. For the same topology the set $A = \{a,b\}$ is $\tau_1$-rg-closed but not $(1,2)$-c*g-closed set.

**EXAMPLE 3.27:** Consider the topological space $X=\{a,b,c\}$ with the topologies $\tau_1 = \{ \emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}\}$ and $\tau_2 = \{ \emptyset, X, \{a\}\}$. Then the set $A = \{a,c\}$ is $(1,2)$-c*g-closed set but not $\tau_1$-gpr-closed. For the same topology the set $A = \{a,b\}$ is $\tau_1$-gpr-closed but not $(1,2)$-c*g-closed set.

**REMARK 3.28:** From the above discussion and known results we have the following diagram.

**EXAMPLE 3.25:** Consider the topological space $X=\{a,b,c\}$ with the topologies $\tau_1 = \{ \emptyset, X, \{c\}, \{a,b\}\}$ and $\tau_2 = \{ \emptyset, X, \{b\}, \{a,b\}, \{b,c\}\}$. Then the set $A = \{a\}$ is $(1,2)$-c*g-closed set but not $\tau_1$-g*-closed. For the same topology the set $A = \{a,b\}$ is $\tau_1$-g*-closed but not $(1,2)$-c*g-closed set.

**REMARK 3.29:** $c^*(i,j)$ is generally not equal to $c^*(j,i)$. Consider the following example.
EXAMPLE 3.30: Consider the topological space $X=\{a,b,c\}$ with the topologies $\tau_1 = \{ \emptyset, X, \{b\}, \{c\}, \{b,c\}\}$ and $\tau_2 = \{ \emptyset, X, \{b\} \}$. Then the set $A = \{b\}$ is $(2,1)$-$c^*$-closed set but $A = \{b\}$ is not $c^*(1,2)$.

REMARK 3.31: If $\tau_1 \subseteq \tau_2 \text{ in } (X, \tau_1, \tau_2)$ then $c^*(2,1) \subseteq c^*(1,2)$. The converse of this remark is not true as seen from the following example.

EXAMPLE 3.32: Let the topological space $X=\{a,b,c\}$ with the topologies $\tau_1 = \{ \emptyset, X, \{a\}, \{b\}, \{a,b\}, \{b,c\}\}$ and $\tau_2 = \{ \emptyset, X, \{b\}, \{c\}, \{b,c\}\}$. In this case $c^*(2,1) \subseteq c^*(1,2)$ but $\tau_1 \not\subseteq \tau_2$.

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REFERENCE


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