International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 2 Issue 6, June - 2013

A Study Of Slow Increasing Functions And Their Applications To Some Sequences Of Integers

K. Santosh Reddy*
K. Madhusudhan Reddy and B. Chandra Sekhar
Vardhaman College of Engineering, Shamshabad, Hyderabad, Andhra Pradesh, India

ABSTRACT

In this article we first define a slow increasing function. We investigate some basic properties of slow increasing function. In addition, several applications in some **some sequences of integers** using the theory of slow increasing functions.

KEYWORDS. Slow Increasing Functions, asymptotically equivalent, sequence of positive integers.

1. INTRODUCTION

Slow increasing functions are defined as follows.

1.1Definition. Let $f:[a,\infty)\to(0,\infty)$ be a continuously differentiable function such that $f'>0$ and $\lim_{x\to\infty} f(x)=\infty$. Then f is said to be a slow increasing function (s.i.f. in short) if $\lim_{x\to\infty}\frac{xf'(x)}{f(x)}=0$

Write $F=\{f : f \text{ is a s.i.f.}\}$.

1.2 Examples. (i) $f(x)=\log x,\ x>1$ is a s.i.f.

Note that $\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\log x=\infty$ and $f'(x)=\frac{1}{x},\forall x>1$ and f' is continuous

Moreover $$\lim_{x\to\infty}\frac{xf'(x)}{f(x)}=\lim_{x\to\infty}\frac{1}{x}\times\frac{x}{\log x}=0$$

(ii) $f(x)=\log\log x, x>e$ is also a s.i.f.

2. SOME PROPERTIES

2.1 Theorem. Let $f,g\in F$ and let $\alpha>0,\ c>0$ be to constants then we have

i) $f+c$ (ii) $f-c$ (iii) cf (iv) fg (v) f^{α} (vi) fog (vii) $\log f$ (viii) $f+g$ all lie in F.

Proof: Given that $f,g\in F$ and $\alpha>0,\ c>0$ be constants.

Proof of (i), (ii), (iii), and (iv) follows the definition 1.1

(v) Let $h=f^{\alpha}$

Note that $\lim_{x\to\infty}h(x)=\lim_{x\to\infty}f(x)^{\alpha}=\infty,$ and $h'(x)=\alpha f(x)^{\alpha-1}f'(x)>0,$ and h' is continuous

Moreover $\lim_{x\to\infty}\frac{xh'(x)}{h(x)}=\lim_{x\to\infty}\frac{x\alpha f(x)^{\alpha-1}f'(x)}{f(x)^{\alpha}}=\alpha\lim_{x\to\infty}\frac{xf'(x)}{f(x)}=0.$ Hence $h=f^{\alpha}\in F$

(vi) Let $h=fog$ i.e $h(x)=f(g(x))$

Note that $\lim_{x\to\infty}h(x)=\lim_{x\to\infty}f(g(x))=\infty,$ and $h'(x)=f'(g(x))g'(x)>0,$ and h' is continuous

Moreover $\lim_{x\to\infty}\frac{xh'(x)}{h(x)}=\lim_{x\to\infty}\frac{xf'(g(x))g'(x)}{f(g(x))}=\lim_{x\to\infty}\frac{g(x)f'(g(x))}{f(g(x))}\times\frac{xg'(x)}{g(x)}=0.$ Hence $h=fog\in F$

(vii) Let $h=\log f$

Note that $\lim_{x\to\infty}h(x)=\lim_{x\to\infty}\log f(x)=\infty,$ and $h'(x)=\frac{f'(x)}{f(x)}>0,$ and h' is continuous

Moreover $\lim_{x\to\infty}\frac{xh'(x)}{h(x)}=\lim_{x\to\infty}\frac{x\frac{f'(x)}{f(x)}}{\log f(x)}=\lim_{x\to\infty}x\frac{f'(x)}{f(x)}\times\frac{1}{\log f(x)}=0.$ Hence $h=\log f\in F$

(viii) Let $h=f+g$

For sufficientely large x, we have $0\le\frac{xf'}{f+g}\le\frac{xf'}{f}$ and $0\le\frac{xg'}{f+g}\le\frac{xg'}{g}$

By adding the above, we get $$0\le\lim_{x\to\infty}\frac{xh'(x)}{h(x)}\le\lim_{x\to\infty}\frac{xf'}{f}+\lim_{x\to\infty}\frac{xg'}{g}=0$$

$$\therefore\lim_{x\to\infty}\frac{xh'(x)}{h(x)}=0 \quad \text{Hence } h=f+g\in F$$

2.2 Theorem. Let $f,g\in F$. Define $h(x)=f(x^\alpha)$ and $k(x)=f(x^\alpha g(x))$ for each x, then $h,k\in F$.

Proof: Given that $f,g\in F$. Define $h(x)=f(x^\alpha)$ and $k(x)=f(x^\alpha g(x))$ for each x.

Let $$h(x)=f(x^\alpha)$$

Note that $\lim_{x\to\infty}h(x)=\lim_{x\to\infty}f(x^\alpha)=\infty,$ and $h'(x)=f'(x^\alpha)\alpha x^{\alpha-1}>0,$ and h' is continuous

Moreover $$\lim_{x\to\infty}\frac{xh'(x)}{h(x)}=\lim_{x\to\infty}\frac{xf'(x^\alpha)\alpha x^{\alpha-1}}{f(x^\alpha)}=\alpha\lim_{x\to\infty}\frac{x^\alpha f'(x^\alpha)}{f(x^\alpha)}=0$$

Hence $h(x)=f(x^\alpha)$ is s.i.f.

Let $k(x)=f(x^\alpha g(x))$ Note that $\lim_{x\to\infty}k(x)=\lim_{x\to\infty}f(x^\alpha g(x))=\infty,$ and

$k'(x)=f'(x^\alpha g(x))\left[\alpha x^{\alpha-1}g(x)+x^\alpha g'(x)\right]>0$ and k' is continuous

Moreover $$\lim_{x\to\infty}\frac{xk'(x)}{k(x)}=\lim_{x\to\infty}\frac{xf'(x^\alpha g(x))\left[\alpha x^{\alpha-1}g(x)+x^\alpha g'(x)\right]}{f(x^\alpha g(x))}$$

$$=\alpha\lim_{x\to\infty}\frac{x^\alpha g(x)f'(x^\alpha g(x))}{f(x^\alpha g(x))}+\lim_{x\to\infty}\frac{x^\alpha g(x)f'(x^\alpha g(x))}{f(x^\alpha g(x))}\times\frac{xg'(x)}{g(x)}=0$$

Therefore $k(x)=f(x^\alpha g(x))$ is s.i.f. Hence $h,k\in F$

2.3 Theorem. Let $f,g\in F$ be such that $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\infty$ and $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]>0.$ Then $\frac{f}{g}\in F.$

Proof: Given that $$f,g\in F,\lim_{x\to\infty}\frac{f(x)}{g(x)}=\infty \text{ and } \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]>0$$

Let $$h(x)=\frac{f(x)}{g(x)} \text{ and } h'(x)=\frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$$

Moreover

$$\lim_{x\to\infty}\frac{xh'(x)}{h(x)}=\lim_{x\to\infty}\frac{x\left(\frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}\right)}{\left(\frac{f(x)}{g(x)}\right)}=\lim_{x\to\infty}\frac{xf'(x)}{f(x)}-\lim_{x\to\infty}\frac{xg'(x)}{g(x)}=0$$

Hence $$\frac{f}{g}\in F$$

2.4 Theorem. Let $h:[a,\infty)\to(0,\infty)$ be a continuously differentiable function such that $h'(x)>0$ and $\lim_{x\to\infty}h(x)=\infty$

(i) Define $g(x) = h(\log x)$. Then $g \in F \Leftrightarrow \lim_{x\to\infty} \frac{h'(x)}{h(x)} = 0$

(ii) Define $k(x) = e^{h(x)}$.Then $k \in F \Leftrightarrow \lim_{x\to\infty} xh'(x) = 0$

Proof: Given that $h'(x) > 0$ and $\lim_{x\to\infty} h(x) = \infty$

(i) Define $g(x) = h(\log x)$ then $g'(x) = \frac{h'(\log x)}{x}$

Suppose $g \in F$ then g satisfies $\lim_{x\to\infty} \frac{xg'(x)}{g(x)} = 0$ i.e. $\lim_{x\to\infty} \frac{x\frac{h'(\log x)}{x}}{h(\log x)} = 0 \Rightarrow \lim_{x\to\infty} \frac{h'(\log x)}{h(\log x)} = 0$

Put $t = \log x$ so that $x \to \infty \Rightarrow t \to \infty \therefore \lim_{t\to\infty} \frac{h'(t)}{h(t)} = 0$ i.e. $\lim_{x\to\infty} \frac{h'(x)}{h(x)} = 0$.

Conversely suppose $\lim_{x\to\infty} \frac{h'(x)}{h(x)} = 0$

Put $t = e^x$ so that $x = \log t$ and $x \to \infty \Rightarrow t \to \infty$ $\Rightarrow \lim_{x\to\infty} \frac{h'(x)}{h(x)} = \lim_{t\to\infty} \frac{h'(\log t)}{h(\log t)} = 0$

Now $\lim_{t\to\infty} \frac{tg'(t)}{g(t)} = \lim_{t\to\infty} \frac{h'(\log t)}{h(\log t)} = \lim_{x\to\infty} \frac{h'(x)}{h(x)} = 0.$ Hence $g \in F$

(ii) Like proof of (i)

2.5 Theorem. If $f \in F$ then $\lim_{x\to\infty} \frac{\log f(x)}{\log x} = 0.$

Proof: Given that $f \in F$, $\lim_{x\to\infty} \frac{\log f(x)}{\log x} = \lim_{x\to\infty} \frac{\left(\frac{f'(x)}{f(x)}\right)}{\left(\frac{1}{x}\right)}$ (byL'Hospital's rule)

$$= \lim_{x\to\infty} \frac{xf'(x)}{f(x)} = 0 \quad \text{i.e. } \lim_{x\to\infty} \frac{\log f(x)}{\log x} = 0.$$

2.6 Theorem. $f \in F$ if and only if to each $\alpha > 0$ there exists x_α such that $\frac{d}{dx}\left[\frac{f(x)}{x^\alpha}\right] < 0, \forall x > x_\alpha$

Proof: We have $\frac{d}{dx}\left[\frac{f(x)}{x^\alpha}\right] = \frac{f'(x)x^\alpha - f(x)\alpha x^{\alpha-1}}{x^{2\alpha}} = \frac{f(x)}{x^{\alpha+1}}\left[\frac{xf'(x)}{f(x)} - \alpha\right]$

Suppose $f \in F$ then $\Rightarrow \lim_{x\to\infty} \frac{xf'(x)}{f(x)} = 0$

i.e. For each $\alpha > 0$ there exists x_α such that $\forall x > x_\alpha$

And $\left|\frac{xf'(x)}{f(x)} - 0\right| < \alpha, \ \forall x > x_\alpha \quad \Rightarrow \frac{d}{dx}\left[\frac{f(x)}{x^\alpha}\right] < 0, \ \forall x > x_\alpha$

To prove the converse assumes that the condition holds.

Let $\alpha > 0$ be given. Then there exist x_α such that $\forall x > x_\alpha$

We have, by hypothesis $\frac{d}{dx}\left[\frac{f(x)}{x^\alpha}\right] < 0$ this implies that $\left|\frac{xf'(x)}{f(x)} - 0\right| < \alpha, \quad \forall x > x_\alpha$

$$i.e. \frac{xf'(x)}{f(x)} \to 0 \text{ as } x \to \infty \Rightarrow \lim_{x\to\infty} \frac{xf'(x)}{f(x)} = 0. \text{ Therefore } f \in F.$$

2.7 Theorem. If $f \in F$ then $\lim_{x\to\infty} \frac{f(x)}{x^{\beta}} = 0$, for all $\beta > 0$

Proof: For any α with $0 < \alpha < \beta$, we get by Theorem **2.6,** $\frac{d}{dx}\left[\frac{f(x)}{x^{\alpha}}\right] < 0, \quad \forall x > x_{\alpha}$ for some x_{α}

This implies that $\frac{f(x)}{x^{\alpha}}$ is decreasing for $x > x_{\alpha}$

Hence $\frac{f(x)}{x^{\alpha}}$ bounded above, say, by M

That is, there exists $M > 0$ such that $0 < \frac{f(x)}{x^{\alpha}} < M, \; \forall x > x_{\alpha}$

$$\lim_{x\to\infty} \frac{f(x)}{x^{\beta}} = \lim_{x\to\infty} \frac{f(x)}{x^{\alpha}} \frac{1}{x^{\beta-\alpha}} = 0$$

2.8 Note. We know that each $f \in F$ is an icreasing function. Moreover by the above theorem it is clear that $\lim_{x\to\infty} \frac{f(x)}{x^{\beta}} = 0, \; \forall \beta > 0$. This shows that the increasing nature of f is slow. That is f does not increase rapidly. This justifies the name given to the members of F.

From the above theorem, we have the following results.

2.9 Theorem. If $f \in F$ then $\lim_{x\to\infty} \frac{f(x)}{x} = 0$ and $\lim_{x\to\infty} f'(x) = 0.$

Proof: In Theorem **2.7** put $\beta = 1$, toget $\lim_{x\to\infty} \frac{f(x)}{x} = 0$.

If $f \in F$, then $\lim_{x\to\infty} \frac{xf'(x)}{f(x)} = 0$

Since $\lim_{x\to\infty} \frac{f(x)}{x} = 0$ we must have $\lim_{x\to\infty} f'(x) = 0.$

2.10 Theorem. Let $f \in F$ then for any $\alpha > -1$ and $\beta \in$ □, the series $\sum_{n=1}^{\infty} n^{\alpha} f(n)^{\beta}$ diverges to ∞.

Proof: We write $\sum_{n=1}^{\infty} n^{\alpha} f(n)^{\beta} = \sum_{n=1}^{\infty} \left(n^{\alpha+1} f(n)^{\beta}\right) \frac{1}{n}$

we know that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to ∞

Given $\alpha > -1 \Rightarrow \alpha + 1 > 0$

If $\beta \geq 0$ then $\lim_{n\to\infty} n^{\alpha+1} f(n)^{\beta} = \infty$

If $\beta > 0$ then $\lim_{n\to\infty} \frac{n}{\left(\frac{f(n)^{-\beta}}{n^{\alpha}}\right)} = \lim_{n\to\infty} \frac{n^{\alpha+1}}{f(n)^{-\beta}} = \infty$ (from Theorem 2.7)

i.e. $\sum_{n=1}^{\infty} n^{\alpha} f(n)^{\beta}$ diverges to ∞

An important byproduct of the above theorem is the following result.

2.11 Theorem. Let $f \in F$. Then for any $\alpha > -1$ and $\beta \in \mathbb{R}$, $\lim\limits_{x\to\infty} \dfrac{\int_a^x t^\alpha f(t)^\beta dt}{\left(\dfrac{x^{\alpha+1} f(x)^\beta}{\alpha+1}\right)} = 1.$

Proof: From Theorem **2.10**, we have $\lim\limits_{n\to\infty} x^{\alpha+1} f(x)^\beta = \infty$

$$\Rightarrow \lim_{x\to\infty} \frac{x^{\alpha+1}}{\alpha+1} f(x)^\beta = \infty, \quad \forall\ \alpha > -1,\ \forall \beta$$

From Theorem **2.10**, we have $\sum\limits_{t=1}^{\infty} t^\alpha f(t)^\beta = \infty \Rightarrow \lim\limits_{x\to\infty} \int_a^x t^\alpha f(t)^\beta dt = \infty$

Consider $\lim\limits_{x\to\infty} \dfrac{\int_a^x t^\alpha f(t)^\beta dt}{\left(\dfrac{x^{\alpha+1} f(x)^\beta}{\alpha+1}\right)} = \lim\limits_{x\to\infty} \dfrac{x^\alpha f(x)^\beta}{x^\alpha f(x)^\beta + \dfrac{x^{\alpha+1}}{\alpha+1} \beta f(x)^{\beta-1} f'(x)}$ (byL'Hospitals's rule)

$$= \lim_{x\to\infty} \frac{x^\alpha f(x)^\beta}{x^\alpha f(x)^\beta \left(1 + \dfrac{\beta}{\alpha+1} \dfrac{xf'(x)}{f(x)}\right)} = 1$$

2.12 Definition. Let $f, g : [a,\infty) \to (0,\infty)$

(i) If $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = 1,$ then f is is said to asymptotically equivalent to g . We describe this by writing $f \sim g$.

(ii) $f = O(g)$ Means $f \le Ag$ for some $A > 0$. In this case we say that f is of large order g.

(iii) $f = o(g)$ Means $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = 0$. In this case we say that f is of small order g.

2.13 Examples. (i) Consider $f(x) = x^n,\ g(x) = x^n + x,$ for all $x > 0$ and $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to\infty} \dfrac{x^n}{x^n + x} = 1$

Therefore $f \sim g$.

(ii) $x = O(10x)$ Because $\dfrac{x}{10x} = \dfrac{1}{10} \Rightarrow x = \dfrac{1}{10}(10x).$

(iii) $x + 1 = o(x^2)$ Becuase $\lim\limits_{x\to\infty} \dfrac{x+1}{x^2} = 0.$

As a result of the Theorem **2.11**, we get the following results as particular cases.

2.14 Theorem. Let $f \in F$. Then we have the following statements.

(i) $\int_a^x f(t)^\beta dt \sim xf(x)^\beta$ (ii) $\int_a^x f(t) dt \sim xf(x)$ (iii) $\int_a^x \dfrac{1}{f(t)} dt \sim \dfrac{x}{f(x)}$

Proof: Let $f \in F$

(i) Put α = 0 in Theorem **2.11**, we get

$$\lim_{x\to\infty} \frac{\int_a^x f(t)^\beta dt}{xf(x)^\beta} = 1 \Rightarrow \int_a^x f(t)^\beta dt \sim xf(x)^\beta$$

(ii) Put α = 0, β = 1 in Theorem **2.11**, we get

$$\lim_{x\to\infty}\frac{\int_a^x f(t)dt}{xf(x)}=1 \Rightarrow \int_a^x f(t)dt \sim xf(x)$$

(iii) Put α = 0, β = -1 in Theorem **2.11**, we get

$$\lim_{x\to\infty}\frac{\int_a^x \frac{1}{f(t)}dt}{\frac{x}{f(x)}}=1 \qquad \Rightarrow \int_a^x \frac{1}{f(t)}dt \sim \frac{x}{f(x)}$$

2.15 Theorem. Let $f \in F$. Then

(i) $\lim_{x\to\infty}\frac{f(x+c)}{f(x)}=1$, For any $c\in\mathbb{R}$ (ii) If $f'(x)$ is decreasing then $\lim_{x\to\infty}\frac{f(cx)}{f(x)}=1$, for any $c\in\mathbb{R}$

Proof: Let $f \in F$

(i) Case (a). Suppose $c>0$

By Lagrange's mean value theorem, There exists a $t\in(x,x+c)$ such that

$$f(x+c)-f(x)=(x+c-x)f'(t) \Rightarrow 0\le\frac{f(x+c)-f(x)}{f(x)}=\frac{cf'(t)}{f(x)}$$

$$\Rightarrow 0\le\lim_{x\to\infty}\frac{f(x+c)-f(x)}{f(x)}=\lim_{x\to\infty}\frac{cf'(t)}{f(x)},\ t\in(x,x+c)$$

$$\Rightarrow \lim_{x\to\infty}\frac{f(x+c)}{f(x)}-1=0,\text{ since }\lim_{x\to\infty}f'(x)=0 \text{ (by Theorem 2.9)}$$

$$\Rightarrow \lim_{x\to\infty}\frac{f(x+c)}{f(x)}=1.$$

Case (b). Suppose $c<0$

By Lagrange's mean value theorem there exists $t\in(x+c,x)$ such that

$$f(x)-f(x+c)=(x-x-c)f'(t) \Rightarrow 0\le\frac{f(x)-f(x+c)}{f(x)}=-\frac{cf'(t)}{f(x)}$$

$$\Rightarrow 0\le\lim_{x\to\infty}\frac{f(x)-f(x+c)}{f(x)}=-c\lim_{x\to\infty}\frac{f'(t)}{f(x)},\ t\in(x+c,x)$$

$$\Rightarrow \lim_{x\to\infty}\frac{f(x+c)}{f(x)}-1=0,\text{ since }\lim_{x\to\infty}f'(x)=0 \quad \text{(by Theorem 2.9)}$$

$$\Rightarrow \lim_{x\to\infty}\frac{f(x+c)}{f(x)}=1.$$

(ii) Case (a). Suppose $c>1$

By Lagrange's mean value theorem there exists $t\in(x,cx)$ such that

$$f(cx)-f(x)=(cx-x)f'(t) \Rightarrow 0\le\frac{f(cx)-f(x)}{f(x)}=\frac{(c-1)xf'(t)}{f(x)}$$

$$\Rightarrow 0\le\lim_{x\to\infty}\frac{f(cx)-f(x)}{f(x)}=(c-1)\lim_{x\to\infty}\frac{xf'(t)}{f(x)},\ t\in(x,cx)$$

And $f(x)$ is decreasing $\Rightarrow f'(x)>f'(t)$

There fore $\lim_{x\to\infty}\frac{f(cx)}{f(x)}-1=0$, since $\lim_{x\to\infty} f'(x)=0$ (by Theorem **2.9**)

$$\Rightarrow \quad \lim_{x\to\infty}\frac{f(cx)}{f(x)}=1.$$

Case (b). Suppose $c<1$

By Lagrange mean value theorem there exists $t\in(cx,x)$ such that

$$f(x)-f(cx)=(x-cx)f'(t) \quad \Rightarrow \quad 0\le\frac{f(x)-f(cx)}{f(x)}=\frac{(1-c)xf'(t)}{f(x)}$$

$$\Rightarrow \quad 0\le\lim_{x\to\infty}\frac{f(x)-f(cx)}{f(x)}=(1-c)\lim_{x\to\infty}\frac{xf'(t)}{f(x)},\ t\in(cx,x)$$

And $f'(x)$ is decreasing $\Rightarrow$ $f'(x)>f'(t)$

There fore $\lim_{x\to\infty}\frac{f(cx)}{f(x)}-1=0$, since $\lim_{x\to\infty} f'(x)=0$ (by Theorem **2.9**)

$$\Rightarrow \lim_{x\to\infty}\frac{f(cx)}{f(x)}=1.$$

2.16 Theorem. Suppose $f\in F$ is such that $f'(x)$ is decreasing. If $0<c_1\le c_2$ and g is a function such that

$$c_1\le g(x)\le c_2 \text{ then } \lim_{x\to\infty}\frac{f(g(x)x)}{f(x)}=1.$$

Proof: Suppose $f\in F$ is such that $f'(x)$ is decreasing

If $0<c_1\le g(x)\le c_2 \Rightarrow f(c_1x)\le f(g(x)x)\le f(c_2x)$ since f is decreasing

$$\Rightarrow \quad \frac{f(c_1x)}{f(x)}\le\frac{f(g(x)x)}{f(x)}\le\frac{f(c_2x)}{f(x)} \Rightarrow \lim_{x\to\infty}\frac{f(c_1x)}{f(x)}\le\lim_{x\to\infty}\frac{f(g(x)x)}{f(x)}\le\lim_{x\to\infty}\frac{f(c_2x)}{f(x)}$$

$$\Rightarrow \quad 1\le\lim_{x\to\infty}\frac{f(g(x)x)}{f(x)}\le 1 \quad \text{(By Theorem 2.15)} \Rightarrow \quad \lim_{x\to\infty}\frac{f(g(x)x)}{f(x)}=1.$$

3. APPLICATIONS OF SLOW INCREASING FUNCTIONS TO SOME SEQUENCES OF INTEGERS

This topic is aimed at applications in some special sequences of positive integers. Infact several asymptotic results related to these integer sequences are derived by using the theory of **Slow Increasing Functions.**

We begin with the following important definition.

Let $f\in F$. Through out this chapter (a_n) denotes a strictly increasing sequence of positive integers such that

$$a_1>1 \text{ And } \lim_{n\to\infty}\frac{a_n}{n^s f(n)}=1 \text{ for some } s\ge 1. \qquad (1)$$

$$\text{i.e. } a_n \sim n^s f(n)$$

There exist several such sequences.

For example $a_n=p_n$, the sequence of prime numbers in increasing order, $f(x)=\log x$ and $s=1$.

By prime number theorem we have $\lim_{n\to\infty}\frac{p_n}{n\log n}=1.$

3.1 Definition. Let (a_n) be asequence as described above. Then for any $x>0$, define $\psi(x)=\sum_{a_n\le x}1$

The number of a_n that do not exceed x.

3.2 Theorem. If (a_n) satisfies (1) and $g \in F$, then

(i) $a_{n+1} \sim a_n$ (ii) $\lim\limits_{n\to\infty} \dfrac{a_{n+1}-a_n}{a_n} = 0$ (iii) $\log a_{n+1} \sim \log a_n$ (iv) $g(a_{n+1}) \sim g(a_n)$

(v) $\log a_n \sim s\log n$ (vi) $\log\log a_n \sim \log\log n$ (vii) $\lim\limits_{x\to\infty} \dfrac{\psi(x)}{x} = 0$

Proof: Let (a_n) satisfies (1) and $g \in F$

(i) Consider $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \dfrac{(n+1)^s f(n+1)}{n^s f(n)}$ $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^s \lim\limits_{n\to\infty} \dfrac{f(n+1)}{f(n)} = 1$ By Theorem **2.15**

$$\Rightarrow \quad a_{n+1} \sim a_n$$

(ii) We have $a_{n+1} \sim a_n \Rightarrow \lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = 1 \Rightarrow \lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} - 1 = 0 \Rightarrow \lim\limits_{n\to\infty} \dfrac{a_{n+1}-a_n}{a_n} = 0$

(iii) Consider $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = 1 \Rightarrow \log\left(\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n}\right) = \log 1 \Rightarrow \lim\limits_{n\to\infty} \log\left(\dfrac{a_{n+1}}{a_n}\right) = 0$

$$\Rightarrow \lim_{n\to\infty}\left(\log a_{n+1} - \log a_n\right) = 0 \Rightarrow \lim_{n\to\infty}\left(\frac{\log a_{n+1} - \log a_n}{\log a_n}\right) = 0$$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{\log a_{n+1}}{\log a_n}\right) = 1 \qquad i.e. \quad \log a_{n+1} \sim \log a_n$$

(iv) As $a_{n+1} \sim a_n$, $g \in F$, we have $\lim\limits_{n\to\infty} \dfrac{g(a_{n+1})}{g(a_n)} = 1 \quad \Rightarrow \; g(a_{n+1}) \sim g(a_n)$

(v) We have $a_n \sim n^s f(n) \Rightarrow \log a_n \sim \log n^s f(n) \Rightarrow \log a_n \sim s\log n + \log f(n)$

$\Rightarrow \dfrac{\log a_n}{s\log n} \sim 1 + \dfrac{\log f(n)}{s\log n} \Rightarrow \lim\limits_{n\to\infty} \dfrac{\log a_n}{s\log n} = 1 + \dfrac{1}{s}\lim\limits_{n\to\infty} \dfrac{\log f(n)}{\log n} \Rightarrow \lim\limits_{n\to\infty} \dfrac{\log a_n}{s\log n} = 1$ By Theorem **2.5**

$$\text{i.e. } \log a_n \sim s\log n$$

(vi) We have $\log a_n \sim s\log n \Rightarrow \log\log a_n \sim \log(s\log n) \Rightarrow \log\log a_n \sim \log s + \log\log n$

$$\Rightarrow \frac{\log\log a_n}{\log\log n} \sim \frac{\log s}{\log\log n} + 1 \Rightarrow \lim_{n\to\infty} \frac{\log\log a_n}{\log\log n} = \log s \lim_{n\to\infty} \frac{1}{\log\log n} + 1$$

$$\Rightarrow \lim_{n\to\infty} \frac{\log\log a_n}{\log\log n} = 1 \qquad \text{i.e. } \log\log a_n \sim \log\log n.$$

(vii) We have $$\psi(x)=\sum_{a_n\le x}1$$

Let n_0 be the largest index such that $a_{n_0}\le x$ then $\psi(x)=n_0 \quad\Rightarrow\quad \dfrac{\psi(x)}{x}=\dfrac{n_0}{x}\le\dfrac{n_0}{a_{n_0}}$

$$\Rightarrow\quad 0\le\lim_{x\to\infty}\frac{\psi(x)}{x}\le\lim_{n_0\to\infty}\frac{n_0}{a_{n_0}}\quad\Rightarrow\quad 0\le\lim_{x\to\infty}\frac{\psi(x)}{x}\le 0\quad\Rightarrow\quad \lim_{x\to\infty}\frac{\psi(x)}{x}=0$$

3.3 Theorem. Suppose (a_n) satisfies (1) and $g\in F$, then for $l>1$,

$$\mathrm{g}(a_n)\sim lg(n)\Leftrightarrow \mathrm{g}(\psi(x))\sim\frac{1}{l}g(x).$$

Proof: First suppose that $$\mathrm{g}(\psi(x))\sim\frac{1}{l}g(x)\quad\Rightarrow \mathrm{g}(\psi(a_n))\sim\frac{1}{l}g(a_n)$$

$$\Rightarrow g(n)\sim\frac{1}{l}g(a_n)\quad\Rightarrow g(a_n)\sim l\mathrm{g}(n).\qquad(\because\psi(a_n)=n)$$

Conversely suppose $$g(a_n)\sim l\mathrm{g}(n)\quad\Rightarrow g(a_n)\sim l\mathrm{g}(\psi(a_n))\quad\Rightarrow\lim_{n\to\infty}\frac{\mathrm{g}(\psi(a_n))}{\frac{1}{l}g(a_n)}=1$$

(2)

If $a_n\le x<a_{n+1}$ we have $\psi(a_n)\le\psi\left(x\right)<\psi\left(a_{n+1}\right)\Rightarrow\quad g(\psi(a_n))\le g(\psi(x))<g(\psi(a_{n+1}))$

since $g\in F$.

And $$g(a_n)\le g\left(x\right)<g\left(a_{n+1}\right)\Rightarrow\quad\frac{1}{l}g(a_n)\le\frac{1}{l}g\left(x\right)<\frac{1}{l}g\left(a_{n+1}\right)\qquad(l\ge1)$$

$$\Rightarrow\lim_{n\to\infty}\frac{g(\psi(a_n))}{\frac{1}{l}g(a_n)}\le\lim_{x\to\infty}\frac{g(\psi(x))}{\frac{1}{l}g(x)}\le\lim_{n\to\infty}\frac{g(\psi(a_n))}{\frac{1}{l}g(a_n)}$$

$$(\because a_{n+1}\sim a_n)\Rightarrow\quad 1\le\lim_{x\to\infty}\frac{g(\psi(x))}{\frac{1}{l}g\left(x\right)}\le1\qquad\text{by }(2)$$

$$\Rightarrow\quad\lim_{x\to\infty}\frac{g(\psi(x))}{\frac{1}{l}g\left(x\right)}=1\qquad\Rightarrow\quad \mathrm{g}(\psi(x))\sim\frac{1}{l}g(x).$$

3.4 Theorem. Suppose (a_n) satisfies (1) and $g\in F$, then (i) $\log a_n\sim s\log n\Leftrightarrow\log\psi(x)\sim\dfrac{1}{s}\log x$

(ii) $\log\log a_n\sim\log logn\Leftrightarrow\log log\psi(x)\sim\log log(x).$

Proof: Given (a_n) satisfies (1) and $g\in F$

(i) In Theorem 3.3 put $g(a_n)=\log a_n$, $g(n)=\log n$ and $l=s$.

And we have $$\log a_n\sim s\log n.$$

(ii) In Theorem 3.3 put $g(a_n) = \log\log a_n$, $g(n) = \log\log n$ and $l = s$

And we have $$\log\log a_n \sim \log\log n.$$

We make use the following well known result in proving theorems to follow.

3.5 Result. Let $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ be two series of positive terms such that $\lim_{n\to\infty} \frac{b_n}{c_n} = 1$. If $\sum_{n=1}^{\infty} c_n$ is divergent

then it is known that $\lim_{n\to\infty} \frac{\sum_{k=1}^{n} b_k}{\sum_{k=1}^{n} c_k} = 1.$

3.6 Theorem. Let $f \in F$, (a_n) satisfies (1) and $f(a_n) \sim f(n)$. Then

$$a_n \sim nf(n) \Leftrightarrow \psi(x) \sim \frac{x}{f(x)} \Leftrightarrow \psi(x) \sim \int_a^x \frac{1}{f(t)}\,dt \Leftrightarrow \sum_{a_k \le x} f(a_k) \sim x.$$

Proof: Given (a_n) satisfies (1) and $f \in F$ and $f(a_n) \sim f(n)$

(3)

First suppose that $\psi(x) \sim \frac{x}{f(x)} \Rightarrow \psi(a_n) \sim \frac{a_n}{f(a_n)} \Rightarrow n \sim \frac{a_n}{f(a_n)}$

$\Rightarrow a_n \sim nf(a_n)$

$$a_n \sim nf(n) \qquad \text{By (3)}$$

Conversely suppose $a_n \sim nf(n) \Rightarrow a_n \sim \psi(a_n) f(n) \Rightarrow \psi(a_n) \sim \frac{a_n}{f(n)}$

$$\Rightarrow \psi(a_n) \sim \frac{a_n}{f(a_n)} \quad \text{by (3)} \quad \Rightarrow \lim_{n\to\infty} \frac{\psi(a_n)}{\frac{a_n}{f(a_n)}} = 1$$

(4)

If $a_n \le x < a_{n+1}$ we have $\psi(a_n) \le \psi(x) < \psi(a_{n+1})$

We have by Theorem **2.6** $\frac{f(x)}{x}$ has negative derivative $\Rightarrow$ $\frac{f(x)}{x}$ is decreasing $\Rightarrow$ $\frac{x}{f(x)}$ is increasing

$$\Rightarrow \frac{a_n}{f(a_n)} \le \frac{x}{f(x)} < \frac{a_{n+1}}{f(a_{n+1})} \Rightarrow \lim_{n\to\infty} \frac{\psi(a_n)}{\frac{a_n}{f(a_n)}} \le \lim_{x\to\infty} \frac{\psi(x)}{\frac{x}{f(x)}} \le \lim_{n\to\infty} \frac{\psi(a_n)}{\frac{a_n}{f(a_n)}} \quad (\because a_{n+1} \sim a_n)$$

$$\Rightarrow 1 \le \lim_{x\to\infty} \frac{\psi(x)}{\frac{x}{f(x)}} \le 1 \quad \text{By (4)} \Rightarrow \lim_{x\to\infty} \frac{\psi(x)}{\frac{x}{f(x)}} = 1 \Rightarrow \psi(x) \sim \frac{x}{f(x)}$$

Suppose $$\psi(x) \sim \frac{x}{f(x)}$$

(5)

Then we have from Theorem **2.14** $$\int_a^x \frac{1}{f(t)}dt \sim \frac{x}{f(x)}$$

(6)

Hence from (5) and (6) $$\psi(x) \sim \int_a^x \frac{1}{f(t)}dt.$$

Also we have from Theorem **2.14** $$\int_a^x f(t)dt \sim xf(x)$$

And since $f(x)$ is increasing, we get $$\sum_{k=1}^{n} f(k) = \int_a^n f(x)dx + h(n) \sim nf(n)$$

(7)

Given that $f(a_n) \sim f(n) \Rightarrow \sum_{k=1}^{n} f(a_k) \sim \sum_{k=1}^{n} f(\mathrm{k})$ By Result 3.5 $\Rightarrow \sum_{k=1}^{n} f(a_k) \sim nf(n)$

By (7)

$$\Rightarrow \sum_{a_k \le a_n} f(a_k) \sim nf(a_n) \Rightarrow \sum_{a_k \le a_n} f(a_k) \sim \psi(a_n) f(a_n) \Rightarrow \lim_{n\to\infty} \frac{\psi(a_n)}{\frac{\sum_{a_k \le a_n} f(a_k)}{f(a_n)}} = 1$$

(8)

If $a_n \le x < a_{n+1}$ we have $$\psi(a_n) \le \psi(x) < \psi(a_{n+1})$$

And $$f(a_n) \le f(x) < f(a_{n+1}) \Rightarrow \sum_{a_k \le a_n} f(a_k) \le \sum_{a_k \le x} f(a_k) < \sum_{a_k \le a_{n+1}} f(a_k)$$

$$\Rightarrow \lim_{n\to\infty} \frac{\psi(a_n)}{\frac{\sum_{a_k \le a_n} f(a_k)}{f(a_n)}} \le \lim_{x\to\infty} \frac{\psi(x)}{\frac{\sum_{a_k \le x} f(a_k)}{f(x)}} \le \lim_{n\to\infty} \frac{\psi(a_n)}{\frac{\sum_{a_k \le a_n} f(a_k)}{f(a_n)}} \qquad (\because a_{n+1} \sim a_n)$$

$$\Rightarrow 1 \le \lim_{x\to\infty} \frac{\psi(x)}{\frac{\sum_{a_k \le x} f(a_k)}{f(x)}} \le 1$$ By (8) $$\Rightarrow \lim_{x\to\infty} \frac{\psi(x)}{\frac{\sum_{a_k \le x} f(a_k)}{f(x)}} = 1 \Rightarrow \psi(x) \sim \frac{\sum_{a_k \le x} f(a_k)}{f(x)}$$

$$\Rightarrow \frac{x}{f(x)} \sim \frac{\sum_{a_k \le x} f(a_k)}{f(x)} \Rightarrow \sum_{a_k \le x} f(a_k) \sim x$$

3.7 Theorem. Let $f \in F$, (a_n) satisfies (1) and $f(a_n) \sim lf(n)$. Then

$$a_n \sim n^s f(n) \Leftrightarrow \psi(x) \sim \frac{l^{\frac{1}{s}} x^{\frac{1}{s}}}{f(x)^{\frac{1}{s}}} \Leftrightarrow \psi(x) \sim \frac{l^{\frac{1}{s}}}{s}\int_a^x \frac{t^{-1+\frac{1}{s}}}{f(t)^{\frac{1}{s}}}dt \Leftrightarrow \sum_{a_k \le x} f(a_k)^{\frac{1}{s}} \sim l^{\frac{1}{s}} x^{\frac{1}{s}}.$$

Proof: Given (a_n) satisfies (1) and $f \in F$ and $f(a_n) \sim lf(n)$

(9)

Suppose $\psi(x) \sim \dfrac{l^{\frac{1}{s}} x^{\frac{1}{s}}}{f(x)^{\frac{1}{s}}} \Rightarrow \psi(a_n) \sim \dfrac{l^{\frac{1}{s}} a_n^{\frac{1}{s}}}{f(a_n)^{\frac{1}{s}}} \Rightarrow n \sim \dfrac{l^{\frac{1}{s}} a_n^{\frac{1}{s}}}{f(a_n)^{\frac{1}{s}}}$

$$\Rightarrow n^s \sim \frac{la_n}{f(a_n)} \Rightarrow a_n \sim n^s f(n)$$ By

(9)

Conversely suppose $a_n \sim n^s f(n) \Rightarrow a_n^{\frac{1}{s}} \sim nf(n)^{\frac{1}{s}} \Rightarrow \lim\limits_{n\to\infty} \dfrac{\psi(a_n)}{\left(\dfrac{l^{\frac{1}{s}} a_n^{\frac{1}{s}}}{f(a_n)^{\frac{1}{s}}}\right)} = 1$ By (9)

(10)

If $a_n \le x < a_{n+1}$ we have $\psi(a_n) \le \psi(x) < \psi(a_{n+1})$

We have by Theorem **2.6** $\dfrac{f(x)}{x}$ has negative derivative $\Rightarrow \dfrac{f(x)}{x}$ is decreasing $\Rightarrow \dfrac{x}{f(x)}$ is increasing

$$\Rightarrow \frac{a_n}{f(a_n)} \le \frac{x}{f(x)} < \frac{a_{n+1}}{f(a_{n+1})} \Rightarrow \lim_{n\to\infty} \frac{\psi(a_n)}{\frac{l^{\frac{1}{s}} a_n^{\frac{1}{s}}}{f(a_n)^{\frac{1}{s}}}} \le \lim_{x\to\infty} \frac{\psi(x)}{\frac{l^{\frac{1}{s}} x^{\frac{1}{s}}}{f(x)^{\frac{1}{s}}}} \le \lim_{n\to\infty} \frac{\psi(a_n)}{\frac{l^{\frac{1}{s}} a_n^{\frac{1}{s}}}{f(a_n)^{\frac{1}{s}}}} \qquad (\because a_{n+1} \sim a_n)$$

$$\Rightarrow 1 \le \lim_{x\to\infty} \frac{\psi(x)}{\frac{l^{\frac{1}{s}} x^{\frac{1}{s}}}{f(x)^{\frac{1}{s}}}} \le 1 \text{ By (10)} \Rightarrow \lim_{x\to\infty} \frac{\psi(x)}{\frac{l^{\frac{1}{s}} x^{\frac{1}{s}}}{f(x)^{\frac{1}{s}}}} = 1 \Rightarrow \psi(x) \sim \frac{l^{\frac{1}{s}} x^{\frac{1}{s}}}{f(x)^{\frac{1}{s}}}$$

$$a_n \sim n^s f(n) \Leftrightarrow \psi(x) \sim \frac{l^{\frac{1}{s}} x^{\frac{1}{s}}}{f(x)^{\frac{1}{s}}} \tag{11}$$

We have from Theorem **2.11** $\lim\limits_{x\to\infty} \dfrac{\int_a^x t^\alpha f(t)^\beta dt}{\frac{x^{\alpha+1} f(x)^\beta}{\alpha+1}} = 1 \Rightarrow \int_a^x t^\alpha f(t)^\beta dt \sim \dfrac{x^{\alpha+1} f(x)^\beta}{\alpha+1}$

In above equation put $\alpha = -1+\frac{1}{s}$ and $\beta = -\frac{1}{s}$ $\Rightarrow$ $\int_a^x t^{-1+\frac{1}{s}} f(t)^{-\frac{1}{s}} dt \square \dfrac{x^{-1+\frac{1}{s}+1} f(x)^{-\frac{1}{s}}}{-1+\frac{1}{s}+1}$

$$\Rightarrow \quad \frac{1}{s}\int_a^x \frac{t^{-1+\frac{1}{s}}}{f(t)^{\frac{1}{s}}} dt \square \frac{x^{\frac{1}{s}}}{f(x)^{\frac{1}{s}}} \qquad \Rightarrow \quad \frac{l^{\frac{1}{s}}}{s}\int_a^x \frac{t^{-1+\frac{1}{s}}}{f(t)^{\frac{1}{s}}} dt \square \frac{l^{\frac{1}{s}} x^{\frac{1}{s}}}{f(x)^{\frac{1}{s}}}$$

(12)

From (11) and (12) $\qquad \psi(x) \square \dfrac{l^{\frac{1}{s}}}{s}\displaystyle\int_a^x \frac{t^{-1+\frac{1}{s}}}{f(t)^{\frac{1}{s}}} dt$

Also we have from Theorem **2.14** $\qquad \displaystyle\int_a^x f(t)dt \square xf(x)$ and $f(x)$ is increasing

Now $\qquad \displaystyle\sum_{k=1}^{n} f(k) = \int_a^n f(x)dx + h(n) \square nf(n)$

(13)

Given that $\qquad f(a)_n \square lf(n) \;\Rightarrow\; \displaystyle\sum_{k=1}^{n} f(a_k) \square l\sum_{k=1}^{n} f(\mathrm{k})$ $\qquad$ By Result 3.5

$\Rightarrow \displaystyle\sum_{k=1}^{n} f(a_k) \square lnf(n)$ By (13) $\Rightarrow \displaystyle\sum_{a_k \le a_n} f(a_k) \square lnf(a_n) \;\Rightarrow\; \sum_{a_k \le a_n} f(a_k) \square l\psi(n) f(a_n)$

$$\Rightarrow \quad \psi(a_n) \square \frac{\sum_{a_k \le a_n} f(a_k)}{lf(a_n)} \qquad \Rightarrow \quad \lim_{n\to\infty} \frac{\psi(a_n)}{\dfrac{\sum_{a_k \le a_n} f(a_k)}{lf(a_n)}} = 1 \qquad (14)$$

If $a_n \le x < a_{n+1}$ we have $\qquad \psi(a_n) \le \psi(x) < \psi(a_{n+1})$

And $f(a_n) \le f(x) < f(a_{n+1}) \Rightarrow lf(a_n) \le lf(x) < lf(a_{n+1})$

$$\Rightarrow \quad \sum_{a_k \le a_n} f(a_k) \le \sum_{a_k \le x} f(a_k) < \sum_{a_k \le a_{n+1}} f(a_k)$$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{\psi(a_n)}{\dfrac{\sum_{a_k \le a_n} f(a_k)}{lf(a_n)}} \le \lim_{x\to\infty} \frac{\psi(x)}{\dfrac{\sum_{a_k \le x} f(a_k)}{lf(x)}} \le \lim_{n\to\infty} \frac{\psi(a_n)}{\dfrac{\sum_{a_k \le a_n} f(a_k)}{lf(a_n)}} \qquad (\because a_{n+1} \square a_n)$$

$$\Rightarrow \quad 1 \le \lim_{x\to\infty} \frac{\psi(x)}{\frac{\sum_{a_k \le x} f(a_k)}{lf(x)}} \le 1 \qquad \text{By (14)} \quad \Rightarrow \quad \lim_{x\to\infty} \frac{\psi(x)}{\frac{\sum_{a_k \le x} f(a_k)}{lf(x)}} = 1$$

$$\Rightarrow \quad \psi(x) \sim \frac{\sum_{a_k \le x} f(a_k)}{lf(x)}$$

$$\Rightarrow \quad \frac{x}{f(x)} \sim \frac{\sum_{a_k \le x} f(a_k)}{lf(x)} \quad \Rightarrow \quad \sum_{a_k \le x} f(a_k) \sim lx \quad \Rightarrow \quad \sum_{a_k \le x} f(a_k)^{\frac{1}{s}} \sim l^{\frac{1}{s}} x^{\frac{1}{s}}.$$

3.8 Theorem. If $g(x)^{\beta}$ is a f.s.i. and $g(a_n) \sim lg(n)$ where (a_n) satisfies (1), then

$$\psi(x) \sim \frac{\sum_{a_k \le x} g(a_k)^{\beta}}{g(x)^{\beta}}, \text{ for all real } \beta.$$

Proof: Given that $g(x)^{\beta}$ is a f.s.i. and $g(a_n) \sim lg(n)$ where (a_n) satisfies (1)

We have from Theorem **2.14** $\quad \int_a^x g(t)^{\beta} dt \sim xg(x)^{\beta}$

And since $g(x)$ is increasing, we get $\quad \sum_{k=1}^{n} g(k)^{\beta} = \int_a^n g(x)^{\beta} dx + h(n) \sim ng(n)^{\beta}$

(15)

Given that $\quad g(a_n) \sim lg(n) \quad \Rightarrow g(a_n)^{\beta} \sim l^{\beta} g(n)^{\beta}$

(16)

$$\Rightarrow \sum_{k=1}^{n} g(a_k)^{\beta} \sim l^{\beta} \sum_{k=1}^{n} g(n)^{\beta} \qquad \text{By Result } \mathbf{3.5}$$

$$\Rightarrow \sum_{k=1}^{n} g(a_k)^{\beta} \sim l^{\beta} ng(n)^{\beta} \quad \text{By (15)} \quad \Rightarrow \sum_{a_k \le a_n} g(a_k)^{\beta} \sim ng(a_n)^{\beta} \quad \text{By (16)}$$

$$\Rightarrow \sum_{a_k \le a_n} g(a_k)^{\beta} \sim \psi(a_n) g(a_n)^{\beta} \quad \Rightarrow \psi(a_n) \sim \frac{\sum_{a_k \le a_n} g(a_k)^{\beta}}{g(a_n)^{\beta}} \quad \Rightarrow \lim_{n\to\infty} \frac{\psi(a_n)}{\frac{\sum_{a_k \le a_n} g(a_k)^{\beta}}{g(a_n)^{\beta}}} = 1 \qquad (17)$$

If $a_n \le x < a_{n+1}$ and $\beta > 0 \quad \Rightarrow \quad \psi(a_n) \le \psi(x) < \psi(a_{n+1})$

We have $g \in F \quad \Rightarrow \quad g(a_n) \le g(x) < g(a_{n+1}) \quad \Rightarrow \quad g(a_n)^{\beta} \le g(x)^{\beta} < g(a_{n+1})^{\beta}$

$$\Rightarrow \sum_{a_k \le a_n} g(a_k)^{\beta} \le \sum_{a_k \le x} g(a_k)^{\beta} < \sum_{a_k \le a_{n+1}} g(a_k)^{\beta}$$

$$\Rightarrow \frac{\sum_{a_k \le a_n} g(a_k)^{\beta}}{g(a_{n+1})^{\beta}} \le \frac{\sum_{a_k \le x} g(a_k)^{\beta}}{g(x)^{\beta}} < \frac{\sum_{a_k \le a_{n+1}} g(a_k)^{\beta}}{g(a_n)^{\beta}}$$

$$\Rightarrow \lim_{n\to\infty} \frac{\psi(a_n)}{\left(\frac{\sum_{a_k \le a_n} g(a_k)^{\beta}}{g(a_n)^{\beta}}\right)} \le \lim_{x\to\infty} \frac{\psi(x)}{\left(\frac{\sum_{a_k \le x} g(a_k)^{\beta}}{g(x)^{\beta}}\right)} \le \lim_{n\to\infty} \frac{\psi(a_n)}{\left(\frac{\sum_{a_k \le a_n} g(a_k)^{\beta}}{g(a_n)^{\beta}}\right)} \qquad (\because a_{n+1} \sim a_n)$$

$$\Rightarrow 1 \le \lim_{x\to\infty} \frac{\psi(x)}{\left(\frac{\sum_{a_k \le x} g(a_k)^{\beta}}{g(x)^{\beta}}\right)} \le 1 \quad \text{By (17)} \quad \Rightarrow \lim_{x\to\infty} \frac{\psi(x)}{\left(\frac{\sum_{a_k \le x} g(a_k)^{\beta}}{g(x)^{\beta}}\right)} = 1$$

$$\Rightarrow \psi(x) \sim \left(\frac{\sum_{a_k \le x} g(a_k)^{\beta}}{g(x)^{\beta}}\right).$$

If $a_n \le x < a_{n+1}$ and $\beta < 0$ $\Rightarrow$ $\psi(a_n) \le \psi(x) < \psi(a_{n+1})$

We have $g \in F$ $\Rightarrow$ $g(a_n) \le g(x) < g(a_{n+1})$ $\Rightarrow$ $g(a_{n+1})^{\beta} \le g(x)^{\beta} < g(a_n)^{\beta}$

$(\because \beta < 0)$

$$\Rightarrow \lim_{n\to\infty} \frac{\psi(a_n)}{\left(\frac{\sum_{a_k \le a_n} g(a_k)^{\beta}}{g(a_n)^{\beta}}\right)} \le \lim_{x\to\infty} \frac{\psi(x)}{\left(\frac{\sum_{a_k \le x} g(a_k)^{\beta}}{g(x)^{\beta}}\right)} \le \lim_{n\to\infty} \frac{\psi(a_n)}{\left(\frac{\sum_{a_k \le a_n} g(a_k)^{\beta}}{g(a_n)^{\beta}}\right)} \qquad (\because a_n \sim a_{n+1})$$

$$\Rightarrow 1 \le \lim_{x\to\infty} \frac{\psi(x)}{\left(\frac{\sum_{a_k \le x} g(a_k)^{\beta}}{g(x)^{\beta}}\right)} \le 1 \quad \text{By (17)} \quad \Rightarrow \lim_{x\to\infty} \frac{\psi(x)}{\left(\frac{\sum_{a_k \le x} g(a_k)^{\beta}}{g(x)^{\beta}}\right)} = 1$$

$$\Rightarrow \psi(x) \sim \frac{\sum_{a_k \le x} g(a_k)^{\beta}}{g(x)^{\beta}}.$$

Hence $$\Rightarrow \psi(x) \sim \frac{\sum_{a_k \le x} g(a_k)^{\beta}}{g(x)^{\beta}} \quad \text{For all } \beta \text{ real.}$$

3.9 Theorem. If (a_n) satisfies (1) and $\lambda(x) = \sum_{p_n \le x} 1$, the number of primes up to x, then $\lambda(x) \sim \psi(x)$.

Proof: The $a_k \le x$ are $a_1, a_2, \ldots, a_{\psi(x)}$.

Let us write $a_k^{\alpha_k} = x, \quad (k = 1, 2, \ldots, \psi(x)) \;\Rightarrow\; \log a_k^{\alpha_k} = \log x \;\Rightarrow\; \alpha_k \log a_k = \log x$

$$\Rightarrow \quad \alpha_k = \frac{\log x}{\log a_k} \qquad (k = 1, 2, \ldots, \psi(x))$$

We know that $$\psi(x) \le \lambda(x) \le \sum_{k=1}^{\psi(x)} [\alpha_k] = \sum_{k=1}^{\psi(x)} \alpha_k = \log x \sum_{k=1}^{\psi(x)} \frac{1}{\log a_k}$$

(18)

From theorem **3.2,** We have $\log a_n \sim s \log n \quad \Rightarrow \quad \dfrac{1}{\log a_n} \sim \dfrac{1}{s \log n}$

$$\Rightarrow \quad \sum_{k=1}^{\psi(x)} \frac{1}{\log a_k} \sim \frac{1}{\log a_1} + \frac{1}{s} \sum_{k=2}^{\psi(x)} \frac{1}{\log k} \qquad \text{By Result 3.5} \qquad (19)$$

We have from Theorem **2.14** $\displaystyle\int_a^x \frac{1}{f(t)} dt \sim \frac{x}{f(x)} \quad \Rightarrow \quad \int_2^x \frac{1}{\log t} dt \sim \frac{x}{\log x}$

(20)

Now $$\frac{1}{\log a_1} + \frac{1}{s} \sum_{k=2}^{\psi(x)} \frac{1}{\log k} = \int_2^{\psi(x)} \frac{1}{\log t} dt + O(1) \sim \frac{\psi(x)}{s \log x}$$ By

(20)

From Equation (19) and above equation $\displaystyle\sum_{k=1}^{\psi(x)} \frac{1}{\log a_k} \sim \frac{\psi(x)}{s \log x}$

From Equation (18) and above equation $\displaystyle\psi(x) \le \lambda(x) \le \frac{h(x)\psi(x)\log x}{s \log x} \qquad (\because h(x) \to 1)$

$$\Rightarrow \quad 1 \le \frac{\lambda(x)}{\psi(x)} \le \frac{h(x)\log x}{s \log x} \qquad \Rightarrow \quad 1 \le \lim_{x\to\infty} \frac{\lambda(x)}{\psi(x)} \le \lim_{x\to\infty} \frac{h(x)\log x}{s \log x}$$

(21)

We have from Theorem **3.4** of (i) $\displaystyle\log \psi(x) \sim \frac{1}{s} \log x \quad \Rightarrow \quad \lim_{x\to\infty} \frac{\log x}{s \log \psi(x)} = 1$

Using this inEquation (21), we get $\displaystyle\Rightarrow \quad 1 \le \lim_{x\to\infty} \frac{\lambda(x)}{\psi(x)} \le 1 \quad (\because h(x) \to 1) \quad \Rightarrow \quad \lim_{x\to\infty} \frac{\lambda(x)}{\psi(x)} = 1$

Hence $$\lambda(x) \sim \psi(x).$$

3.10 Theorem. Let $f \in F, (a_n)$ satisfies (1).

Then (i) $\displaystyle\sum_{k=1}^{n} a_k^\alpha \sim \frac{n^{s\alpha+1} f(n)^\alpha}{s\alpha + 1} \sim \frac{n}{s\alpha + 1} a_n^\alpha \qquad (\alpha > 0)$ (ii)

$$\sum_{a_n \le x} a_n^\alpha \sim \frac{\psi(x)}{s\alpha + 1} x^\alpha \qquad (\alpha > 0)$$

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 2 Issue 6, June - 2013

Proof: Given that Let $f \in F, \left(a_n\right)$ satisfies (1)

(i) Let us consider the sum $$1+2+...+(n'-1)+\sum_{k=n'}^{n}(k^s f(k)^s)^\alpha$$

(22)

Where n' is positive integer in the interval $\left[a,\infty\right)$

From Equation (1) we have $a_n \sim n^s f(n) \quad \Rightarrow \quad a_n^\alpha \sim n^{s\alpha} f(n)^\alpha \; (\because \alpha > 0)$

(23)

We know that $x^s f(x)^s$ is increasing $\Rightarrow \displaystyle\sum_{k=n'}^{n}(k^s f(k)^s)^\alpha = \int_{n'}^{n} x^{s\alpha} f(x)^\alpha dx + \mathrm{O}(n^{s\alpha} f(n)^\alpha$

(24)

From Theorem 2.13, we have $$\lim_{x\to\infty}\frac{\int_a^x t^\alpha f(t)^\beta dt}{\left(\dfrac{x^{\alpha+1} f(x)^\beta}{\alpha+1}\right)} = 1 \quad \Rightarrow \quad \int_a^x t^\alpha f(t)^\beta dt \sim \frac{x^{\alpha+1} f(x)^\beta}{\alpha+1}$$

Put $\alpha = s\alpha$ and $\beta = \alpha$ in above equation, we get $\displaystyle\int_{n'}^{n} x^{s\alpha} f(x)^\alpha dx \sim \frac{n^{s\alpha+1} f(n)^\alpha}{s\alpha+1}$

(25)

From Eqations (22), (24) and (25), we get

$$1+2+...+(n'-1)+\sum_{k=n'}^{n}(k^s f(k)^s)^\alpha \sim \frac{n^{s\alpha+1} f(n)^\alpha}{s\alpha+1} = \frac{n^{s\alpha} n f(n)^\alpha}{s\alpha+1}) \tag{26}$$

$$\Rightarrow \quad 1+2+...+(n'-1)+\sum_{k=n'}^{n}(k^s f(k)^s)^\alpha \sim \frac{n a_n^\alpha}{s\alpha+1} \qquad \text{By (23)} \tag{27}$$

FromEquations (23), (26) and (27) and using Result 3.5, we get

$$\sum_{k=1}^{n} a_k^\alpha \sim \frac{n^{s\alpha+1} f(n)^\alpha}{s\alpha+1} \sim \frac{n}{s\alpha+1} a_n^\alpha \qquad (\alpha > 0)$$

(ii) If $a_n \le x < a_{n+1}$ and $\alpha > 0 \quad \Rightarrow \quad \psi(a_n) \le \psi\left(x\right) < \psi\left(a_{n+1}\right)$

And $\quad a_n^\alpha \le x^\alpha < a_{n+1}^\alpha \quad \Rightarrow \quad (s\alpha+1)\displaystyle\sum_{a_k \le a_n} a_k^\alpha \le (s\alpha+1)\sum_{a_k \le x} a_k^\alpha < (s\alpha+1)\sum_{a_k \le a_{n+1}} a_k^\alpha$

$$\Rightarrow \lim_{n\to\infty}\frac{\psi(a_n)}{\left(\dfrac{(s\alpha+1)\sum_{a_k \le a_n} a_k^\alpha}{a_n^\alpha}\right)} \le \lim_{x\to\infty}\frac{\psi(x)}{\left(\dfrac{(s\alpha+1)\sum_{a_k \le x} a_k^\alpha}{x^\alpha}\right)} \le \lim_{n\to\infty}\frac{\psi(a_n)}{\left(\dfrac{(s\alpha+1)\sum_{a_k \le a_n} a_k^\alpha}{a_n^\alpha}\right)} \qquad (\because a_n \sim a_{n+1})$$

(28)

We have $$\sum_{k=1}^{n} a_k^\alpha \sim \frac{n a_n^\alpha}{s\alpha+1} \Rightarrow \sum_{a_k \le a_n} a_k^\alpha \sim \frac{\psi(n) a_n^\alpha}{s\alpha+1} \Rightarrow \psi(a_n) \sim \frac{s\alpha+1 \sum_{a_k \le a_n} a_k^\alpha}{a_n^\alpha}$$

$$\Rightarrow \lim_{n\to\infty} \frac{\psi(a_n)}{\frac{s\alpha+1 \sum_{a_k \le a_n} a_k^\alpha}{a_n^\alpha}} = 1$$

Equation (28) implies $$1 \le \lim_{x\to\infty} \frac{\psi(x)}{\left(\frac{(s\alpha+1)\sum_{a_k \le x} a_k^\alpha}{x^\alpha}\right)} \le 1 \Rightarrow \lim_{x\to\infty} \frac{\psi(x)}{\left(\frac{(s\alpha+1)\sum_{a_k \le x} a_k^\alpha}{x^\alpha}\right)} = 1$$

$$\Rightarrow \psi(x) \sim \frac{(s\alpha+1)\sum_{a_k \le x} a_k^\alpha}{x^\alpha} \Rightarrow \sum_{a_n \le x} a_n^\alpha \sim \frac{\psi(x)}{s\alpha+1} x^\alpha. \quad (\alpha > 0).$$

We apply the results discussed in this article to look into some of the applications in number theory.

Acknowledgments. The author would like to thank the anonymous referees for a careful reading and evaluating the original version of this manuscript.

REFERENCES

[1] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Fourth Edition, 1960.

[2] R. Jakimczuk. *Functions of slow increase and integer sequences*, Journal of Integer Sequences, Vol. 13 (2010), Article 10.1.1.

[3] R. Jakimczuk. *A note on sums of powers which have a fixed number of prime factors*, J.Inequal. Pure Appl. Math. 6 (2005), 5-10.

[4] R. Jakimczuk. *The ratio between the average factor in a product and the last factor*, mathematical sciences: Quarterly Journal 1 (2007), 53-62.

[5] Y. Shang, *On a limit for the product of powers of primes*, Sci. Magna **7** (2011) 31-33.

[6] J. Rey Pastor, P. Pi Calleja and C. Trejo, *An alisis Matem arico*, Volumen I, Octava Edicion, Editorial Kapelusz, 1969.