A Related Fixed Point Theorem of Integral Type on Two Fuzzy 2-Metric Spaces

Pheiroijam Suranjay Singh
Sagolband Takyel Kolom Leikai
Imphal West, Manipur - 795001, India.

Abstract

In this paper, a related fixed point theorem is obtained. It extends a result proved by R.K. Namdeo, N.K. Tiwari, B. Fisher and K. Tas [9]. The notion of fuzzy 2-metric spaces satisfying integral type inequalities is used.

Keywords: Fuzzy 2-metric space, fixed point, related fixed point, integral type inequality.

2000 AMS Subject Classification: 47H10, 54H25.

Introduction

The concept of fuzzy sets was introduced by L. Zadeh [14] in 1965. Fuzzy metric space was introduced by Kramosil and Michalek [7] in 1975. Then, it was modified by George and Veeramani [4] in 1994. Fuzzy has been studied and developed by many mathematicians for many years. Introduction of fuzzy 2-metric space is one of such developments. Gahler [10, 11] investigated 2-metric spaces in a series of his papers. Fuzzy 2-metric space is studied in [6, 8, 12, 13] and many others. Related fixed point is studied in [1, 2, 3, 5, 9] and many more.

Some definitions are stated as follows:

Definition 1.1: A binary operation \( \ast : [0, 1] \times [0, 1] \rightarrow [0, 1] \) is called a \( t \)-norm in \(( [0, 1], \ast ) \) if following conditions are satisfied:
For all \( a, b, c, d \in [0, 1] \),

i. \( a \ast 1 = a \),

ii. \( a \ast b = b \ast a \),

iii. \( a \ast b \leq c \ast d \) whenever \( a \leq c \) and \( b \leq d \),

iv. \( a \ast (b \ast c) = (a \ast b) \ast c \).

Definition 1.2. The 3-tuple \(( X, \mu, \ast )\) is called a fuzzy 2-metric space if \( X \) is an arbitrary set, \( \ast \) is a continuous \( t \)-norm and \( \mu \) is a fuzzy set in \( X \times [0, \infty) \) satisfying the following conditions:
For all \( x, y, z, u \in X \) and \( t_1, t_2, t_3 > 0 \),

i. \( \mu (x, y, z, 0) = 0 \),

ii. \( \mu (x, y, z, t) = 1, t > 0 \) and when at least two of the three points are equal,

iii. \( \mu (x, y, z, t) = \mu (y, x, z, t) = \mu (z, x, y, t) \) (symmetry about three variables),

iv. \( \mu (x, y, z, t_1 + t_2 + t_3) \geq \mu (x, y, u, t_1) \ast \mu (x, u, z, t_2) \ast \mu (u, y, z, t_3) \),

v. \( \mu(x, y, z, \cdot) : [0, \infty) \rightarrow [0, 1] \) is left continuous,
vi. \( \lim_{t \to \infty} \mu(x, y, z, t) = 1 \).

**Definition 1.3**: Let \((X, \mu, *)\) be a fuzzy 2-metric space. A sequence \(\{x_n\}\) in \(X\) is said to:

i. converge to \(x\) in \(X\) if and only if \(\lim_{n \to \infty} \mu(x_n, x, a, t) = 1\) \(\forall a \in X\) and \(t > 0\).

ii. be a Cauchy sequence if and only if \(\lim_{n \to \infty} \mu(x_{n+p}, x_n, a, t) = 1\) \(\forall a \in X\), \(p > 0\) and \(t > 0\).

**Definition 1.4**: A fuzzy 2-metric space \((X, \mu, *)\) is said to be complete if and only if every Cauchy sequence in \(X\) is convergent in \(X\).

The following was proved in [9].

**Theorem 1.1**: Let \((X, d)\) and \((Y, \rho)\) be complete metric spaces. Let \(T\) be a mapping of \(X\) into \(Y\) and \(S\) be a mapping of \(Y\) into \(X\) satisfying the inequalities

\[
d(Sy, Sy') \leq c \max \{d(Sy, Sy'), \rho(Tx, Tx'), d(x, x'), d(Sy, Sy') \}
\]

\[
\rho(Tx, Tx') \leq c \max \{d(Sy, Sy'), \rho(Tx, Tx'), d(x, x'), \rho(y, y') \}
\]

for all \(x, x' \in X\) and \(y, y' \in Y\), where \(0 \leq c < 1\). If either \(S\) or \(T\) is continuous, then \(ST\) has a unique fixed point \(z\) in \(X\) and \(TS\) has a unique fixed point \(w\) in \(Y\). Further, \(Tz = w\) and \(Sw = z\).

Now, theorem 1.1 is extended to two pairs of mappings in integral and fuzzy 2-metric space settings as follows.

**Main result**

**Theorem 2.1**: Let \((X, \mu, a, t)\) and \((Y, \nu, a, t)\) be two complete fuzzy 2-metric spaces. Let \(A, B\) be mappings of \(X\) into \(Y\) and \(S, T\) be mappings of \(Y\) into \(X\) satisfying the inequalities

\[
\min \{\mu(Sy, Ty', a, t) \nu(Ax, Bx', a, t), \mu(x, Sy, a, t) \nu(y, Ax, a, t), \mu(x, x', a, t) \mu(Sy, Ty', a, t) \}
\]

\[
\int_1^k \phi(s) ds \geq \int_1^\infty \phi(s) ds
\]

(1)

\[
\min \{\nu(Ax, Bx', a, t) \nu(As, B', a, t), \nu(y, y', a, t) \nu(Ax, Bx', a, t), \nu(y, y', a, t) \nu(x, x', a, t) \}
\]

\[
\int_1^k \phi(s) ds \geq \int_1^\infty \phi(s) ds
\]

(2)

for all \(x, x' \in X\) and \(y, y' \in Y\), where \(k \in (0, 1)\). If \(A\) and \(S\) or \(B\) and \(T\) are continuous, then \(SA\) and \(TB\) have a unique common fixed point \(z\) in \(X\) and \(BS\) and \(AT\) have unique common fixed point \(w\) in \(Y\). Further, \(Az = Bz = w\) and \(Sw = Tw = z\).

**Proof**: Let \(x\) be an arbitrary point in \(X\). We define sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) and \(Y\) respectively as:

\(Sx_{2n-1} = x_{2n-1}, \quad Bx_{2n-1} = y_{2n}, \quad Ty_{2n} = x_{2n}, \quad Ax_{2n} = y_{2n-1}\), for \(n = 1, 2, 3, \ldots\)
Applying inequality (1), we get

\[ \int_{1}^{\infty} k\mu(Sy_{2n-1}, Ty_{2n}, x, a, t) \mu(SA_{2n}, TB_{2n-1}, a, t) \varphi(s) ds \]

\[ = \int_{1}^{\infty} k^2 \mu(x_{2n-1}, x_{2n}, a, t) \varphi(s) ds \]

\[ \min \{ \mu(Sy_{2n-1}, Ty_{2n}, x, a, t) v(Ax_{2n}, Bx_{2n-1}, a, t), \mu(x_{2n-1}, y_{2n-1}, a, t) v(y_{2n-1}, Ax_{2n}, a, t), \mu(x_{2n-1}, x_{2n-1}, a, t) v(y_{2n-1}, y_{2n-1}, a, t) \} \]

\[ \mu(x_{2n-1}, x_{2n}, a, t) \mu(Sy_{2n-1}, Ty_{2n}, a, t), \mu(Sy_{2n-1}, SA_{2n}, a, t) \mu(Ty_{2n}, TB_{2n-1}, a, t) \} \]

\[ \geq \int_{1}^{\infty} \mu(x_{2n-1}, x_{2n}, a, t) \varphi(s) ds \]

from which it follows that

\[ \int_{1}^{\infty} k\mu(x_{2n-1}, x_{2n}, a, t) \varphi(s) ds \geq \int_{1}^{\infty} \min \{ v(y_{2n-1}, y_{2n}, a, t), \mu(x_{2n-1}, x_{2n}, a, t) \} \varphi(s) ds \]

(3)

Applying inequality (2), we get

\[ \int_{1}^{\infty} k\mu(ax_{2n-1}, Bx_{2n-1}, a, t) v(BS_{2n-1}, AT_{2n}, a, t) \varphi(s) ds \]

\[ = \int_{1}^{\infty} k^2 \mu(y_{2n-1}, y_{2n}, a, t) \varphi(s) ds \]

\[ \min \{ \mu(Sy_{2n-1}, Ty_{2n}, a, t) v(Ax_{2n}, Bx_{2n-1}, a, t), \mu(x_{2n-1}, y_{2n-1}, a, t) v(y_{2n-1}, Ax_{2n}, a, t), \mu(x_{2n-1}, x_{2n-1}, a, t) v(y_{2n-1}, y_{2n-1}, a, t) \} \]

\[ \mu(x_{2n-1}, y_{2n-1}, a, t) v(y_{2n-1}, Bx_{2n-1}, a, t), v(Ax_{2n}, BS_{2n-1}, a, t) v(Bx_{2n-1}, AT_{2n}, a, t) \} \]

\[ \geq \int_{1}^{\infty} \mu(y_{2n-1}, y_{2n}, a, t) \varphi(s) ds \]

from which it follows that

\[ \int_{1}^{\infty} k\mu(y_{2n-1}, y_{2n}, a, t) \varphi(s) ds \geq \int_{1}^{\infty} \min \{ v(y_{2n-1}, y_{2n}, a, t), \mu(x_{2n-1}, x_{2n}, a, t) \} \varphi(s) ds \]

(4)

(3) and (4) can be written as

\[ \int_{1}^{\infty} k\mu(x_{n-1}, x_{n}, a, t) \varphi(s) ds \geq \int_{1}^{\infty} \min \{ v(y_{n-1}, y_{n}, a, t), \mu(x_{n-1}, x_{n}, a, t) \} \varphi(s) ds \]
\[
\int_1^k \mu\left(y_{n-1}, y_n, a, t\right) \phi(s) \, ds \geq \int_1^k \min\left\{ \nu\left(y_{n-1}, y_n, a, t\right), \mu\left(x_{n-1}, x_n, a, t\right) \right\} \phi(s) \, ds
\]

which can be again written as

\[
\int_1^k \mu\left(x_{n+1}, x_n, a, t\right) \phi(s) \, ds \geq \int_1^k \min\left\{ \nu\left(y_{n+1}, y_n, a, t\right), \mu\left(x_{n+1}, x_n, a, t\right) \right\} \phi(s) \, ds
\]  

(5)

\[
\int_1^k \nu\left(y_{n+1}, y_n, a, t\right) \phi(s) \, ds \geq \int_1^k \min\left\{ \nu\left(y_{n+1}, y_n, a, t\right), \mu\left(x_{n+1}, x_n, a, t\right) \right\} \phi(s) \, ds
\]  

(6)

From (5) and (6), by induction, we get

\[
\int_1^k \mu\left(x_{n+1}, x_n, a, t\right) \phi(s) \, ds \geq \int_1^k \frac{1}{k^n} \min\left\{ \nu\left(y_1, y_2, a, t\right), \mu(x_1, x_2, a, t) \right\} \phi(s) \, ds
\]

\[
\int_1^k \nu\left(y_{n+1}, y_n, a, t\right) \phi(s) \, ds \geq \int_1^k \frac{1}{k^n} \min\left\{ \nu\left(y_1, y_2, a, t\right), \mu(x_1, x_2, a, t) \right\} \phi(s) \, ds
\]

Let \( t_1 = \frac{t}{p} \). Now,

\[
\int_1^k \mu(x_n, x_{n+p}, a, t) \phi(s) \, ds = \int_1^k \mu(x_n, x_{n+p}, a, t + \cdots + p \text{ times}) \phi(s) \, ds
\]

\[
\geq \int_1^k \frac{1}{k^n} \min\left\{ \nu(y_1, y_2, a, t), \mu(x_1, x_2, a, t) \right\} \phi(s) \, ds \times \int_1^k \frac{1}{k^{n+1}} \min\left\{ \nu(y_1, y_2, a, t), \mu(x_1, x_2, a, t) \right\} \phi(s) \, ds
\]

which implies that

\[
\lim \int_1^k \mu(x_n, x_{n+p}, a, t) \phi(s) \, ds \geq 1
\]

\[
\Rightarrow \mu(x_n, x_{n+p}, a, t) \geq 1
\]

\[\Rightarrow \{ x_n \} \text{ is a Cauchy sequence with a limit } z \text{ in } X.\]

Similarly, \( \{ y_n \} \) is a Cauchy sequence with a limit \( w \) in \( Y \).

Now, on using the continuity of \( A \) and \( S \) respectively, we get
\[ w = \lim_{n \to \infty} y_{2n-1} = \lim Ax_{2n} = Az \quad \text{and} \quad z = \lim_{n \to \infty} x_{2n} = \lim Sy_{2n} = Sw \]

so that we get

\[ Az = w \] (7)

\[ Sw = z \] (8)

From (7) and (8), we get

\[ SAz = z \] (9)

Again applying inequality (1), we get

\[
\int_{1}^{k\mu(SAx_{2n}, TBx_{2n-1}, a, t)} \varphi(s) ds \\
\geq \int_{1}^{\min\{\nu(Ax_{2n}, Bx_{2n-1}, a, t), \nu(y_{2n}, Ax_{2n}, a, t), \mu(x_{2n-1}, x_{2n}, a, t)\}} \varphi(s) ds
\]

(10)

On letting \( n \to \infty \), we have

\[
\int_{1}^{k\mu(Sw, TBz, a, t)} \varphi(s) ds \geq \int_{1}^{\nu(Az, w, a, t)} \varphi(s) ds
\]

By (7), we have

\[
\int_{1}^{k\mu(Sw, TBz, a, t)} \varphi(s) ds \geq 0
\]

\[ \Rightarrow k\mu(Sw, TBz, a, t) \geq 1 \]

which implies that

\[ Sw = TBz \]

and from (8), we get

\[ z = TBz \] (11)

From (9) and (11), we get

\[ SAz = z = TBz \] (12)

Now, (10) gives

\[
\int_{1}^{k\mu(x_{2n-1}, Ty_{2n}, a, t)} \varphi(s) ds
\]
\[
\min \{ \nu(x_{2n}^{-1}, Bx_{2n}^{-1}, a, t), \nu(y_{2n}^{-1}, Ax_{2n}^{-1}, a, t), \mu(x_{2n}^{-1} y_{2n}^{-1}, a, t) \} \\
\geq \int_1^\infty \phi(s) \, ds
\]

On letting \( n \to \infty \), we get

\[
\int_1^\infty k \mu(z, Tw, a, t) \phi(t) \, dt \geq 0
\]

\[
\Rightarrow k \mu(z, Tw, a, t) \geq 1
\]

which implies that

\[
z = Tw
\]

Again, applying inequality (2), we get

\[
\int_1^\infty k \nu(BSy_{2n}^{-1}, ATy_{2n}^{-1}, a, t) \phi(s) \, ds
\]

\[
\min \{ \mu(Sy_{2n}^{-1}, Ty_{2n}^{-1}, a, t), \mu(x_{2n}^{-1} y_{2n}^{-1}, a, t), \nu(y_{2n}^{-1} y_{2n}^{-1}, a, t), \nu(Ax_{2n}^{-1} Bx_{2n}^{-1}, a, t) \} \\
\geq \int_1^\infty \phi(s) \, ds
\]

On letting \( n \to \infty \), we get

\[
\int_1^\infty k \nu(BSw, ATw, a, t) \phi(s) \, ds \geq 0
\]

\[
\Rightarrow k \nu(BSw, ATw, a, t) \geq 1
\]

which implies that

\[
BSw = ATw
\]

Now, (14) gives

\[
\int_1^\infty k \nu(y_{2n}^{-1}, ATy_{2n}^{-1}, a, t) \phi(s) \, ds
\]

\[
\min \{ \mu(Sy_{2n}^{-1}, Ty_{2n}^{-1}, a, t), \mu(x_{2n}^{-1} y_{2n}^{-1}, a, t), \nu(y_{2n}^{-1} y_{2n}^{-1}, a, t), \nu(Ax_{2n}^{-1} Bx_{2n}^{-1}, a, t) \} \\
\geq \int_1^\infty \phi(s) \, ds
\]

On letting \( n \to \infty \), we get

\[
\int_1^\infty k \nu(w, ATw, a, t) \phi(s) \, ds \geq 0
\]
\[ \Rightarrow k\nu(w, ATw, a, t) \geq 1 \]

which implies that
\[ w = ATw \] (16)

From (15) and (16), we get
\[ BSw = w = ATw \] (17)

From (8) and (17), we get
\[ Bz = w \] (18)

From (7) and (18), we get
\[ Az = Bz = w \] (19)

From (8) and (13), we get
\[ Sw = Tw = z \] (20)

Similarly, on using the continuity of \( B \) and \( T \), the above results hold.

To prove the uniqueness, let \( SA \) and \( TB \) have a second distinct common fixed point \( z' \) in \( X \) and \( BS \) and \( AT \) have a second distinct common fixed point \( w' \) in \( Y \).

Applying inequality (1), we have
\[
\int_1^t k\mu(z, z', a, t) \varphi(s) ds \\
\geq \int_1^t \min\{\mu(z, z', a, t) \nu(Az, Bz', a, t), \mu(z', z', a, t) \nu(Bz', Az, a, t), \mu(z, z, a, t) \} \varphi(s) ds \\
\Rightarrow \int_1^t k\mu(z, z', a, t) \varphi(s) ds \geq \int_1^t \min\{\nu(Az, Bz', a, t), (Bz', Az, a, t)\} \varphi(s) ds \\
\Rightarrow \int_1^t k\mu(z, z', a, t) \varphi(s) ds \geq \int_1^t \nu(Az, Bz', a, t) \varphi(s) ds \\
\] (21)

Applying inequality (2), we get
\[
\int_1^t k\nu^2(Az, Bz', a, t) \varphi(s) ds \\
\]
\[\min\{\mu(z, z', a, t)\upsilon(Az, Bz', a, t), \mu(z', z, a, t)\upsilon(Bz, Az, a, t)\} \geq \int_1^k \upsilon(Az, Bz', a, t)\upsilon(Az, Bz', a, t) + \upsilon(Az, Bz, a, t)\upsilon(Bz', Az, a, t)\} \varphi(s) ds\]

\[\Rightarrow \int_1^k \upsilon(Az, Bz', a, t) \varphi(s) ds \geq \int_1^k \mu(z, z', a, t) \varphi(s) ds\]  \hspace{1cm} (22)

From (21) and (22), we get

\[\int_1^k \mu(z, z', a, t) \varphi(s) ds \geq \int_1^k \mu(z, z', a, t) \varphi(s) ds\]

\[\Rightarrow \int_1^k \mu(z, z', a, t) \varphi(s) ds \geq \int_1^k \mu(z, z', a, t) \varphi(s) ds \geq \ldots \geq \int_1^k \mu(z, z', a, t) \varphi(s) ds\]

\[\Rightarrow \mu(z, z', a, t) \geq 1\]

which implies that

\[z = z'.\]

This proves the uniqueness of \(z\). Similarly, the uniqueness of \(w\) can be proved.

The following corollary is a fuzzy 2-metric space version of theorem 1.1 in integral setting.

**Corollary 2.2**: Let \((X, \mu, a, t)\) and \((Y, \nu, a, t)\) be two complete fuzzy 2-metric spaces. Let \(S\) be mappings of \(X\) into \(Y\) and \(T\) be mappings of \(Y\) into \(X\) satisfying the inequalities

\[\min\{\mu(Ty, Ty', a, t)\upsilon(Sx, Sx', a, t), \mu(x', Ty, a, t)\upsilon(y', Sx, a, t), \mu(x, x', a, t)\upsilon(Ty, Ty', a, t), \mu(Ty, Ty', a, t)\upsilon(Ty, Ty', a, t)\} \varphi(s) ds\]

\[\int_1^k \mu(Ty, Ty', a, t) \mu(TSx, TSx', a, t) \varphi(s) ds \geq \int_1^k \mu(Ty, Ty', a, t) \mu(TSx, TSx', a, t) \varphi(s) ds\]

\[\min\{\mu(Ty, Ty', a, t)\upsilon(Sx, Sx', a, t), \mu(x', Ty, a, t)\upsilon(y', Sx, a, t), \upsilon(y, y', a, t)\upsilon(Sx, Sx', a, t), \upsilon(Sx, STy, a, t)\upsilon(Sx, STy', a, t)\} \varphi(s) ds\]

\[\int_1^k \upsilon(Sx, Sx', a, t) \upsilon(STy, STy', a, t) \varphi(s) ds \geq \int_1^k \upsilon(Sx, Sx', a, t) \upsilon(STy, STy', a, t) \varphi(s) ds\]
for all $x, x'$ in $X$ and $y, y'$ in $Y$, where $k \in (0, 1)$. If either $S$ or $T$ is continuous, then $TS$ has a unique fixed point $z$ in $X$ and $ST$ has a unique fixed point $w$ in $Y$. Further, $Sz = w$ and $Tw = z$.

**Proof:** By putting $A = B = S$ and $S = T = T$ in theorem 2.1, the result easily follows.

**References**


