A Parametric Approximation for the Radius of Curvature of a Bimetallic Strip
(An Immediate Approximation to the Timoshenko Formula)

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Abstract—A parametric approach has been used to derive an approximate formula for the prediction of the radius of curvature of a thin bimetallic strip that is initially flat at ambient temperature, is both flat and straight, but at above ambient temperature, forms into an arc of a circle. The formula enables the evaluation of the radius of curvature of the strip as a function of heating or cooling. A formula for calculating the radius of curvature of a bimetallic strip already exists, and was produced by Timoshenko in his paper on Bimetallic Thermostats. The formula by Timoshenko has been vigorously tested, tried and proven and accepted in countless papers and journals since its original publication. The parametric approach solution introduced in this work gives an approximate solution to the Timoshenko formula for equal thicknesses of two mating metals within the bimetallic. The Khathkate Singh Mirchandani (KSM) formula presented here is taken from the first order approximation derived by Angel and Haritos by incorporating the ratio of Young's modulus of the bimetallic materials. The simulation results and the overall close agreement with the Timoshenko formula have been put forward here. Also, the solution derived by the authors Khathkate et al. shows better prediction as compared to the solutions derived by other researchers.

Keywords—bimetallic strip; INVAR 36; Timoshenko; low coefficient of expansion

I. MOTIVATION

The original Timoshenko [1] bimetallic bending formula was published in 1925 and since then has been extensively applied by multitudes of engineers and scientists and referred to in many papers such as by Krulevitch [2], Prasad [3] and books Kanthal [4]. Whilst it has been proven and accepted to be the formula to evaluate the radius of curvature of an initially flat bimetallic strip, it is unwieldy and a complex formula to evaluate. This work introduces a new simpler, quicker method to evaluate the radius of curvature of the bimetallic strip from an initially flat ambient condition that has been uniformly heated. When a bimetallic strip is uniformly heated along its entire length, it will bend or deform into an arc of a circle with the radius of curvature, the value of which, is dependent on the geometry and metal components making up the strip. Also, the nature of the bend is a function of temperature change from the ambient temperature which is characteristically asymptotic. The new formula, introduced here, is taken from the first order approximation by Angel and Haritos with suitable incorporation of the ratio of Young's modulus of elasticity of the two materials that form the bimetallic.

The new formula introduced here, closely approximates to the Timoshenko formula with the exception of accommodating the change in the thicknesses of the two mating metals making up the bimetallic strip. It is important to note that in the majority of applications of bimetallic strip, the ratio of the thickness of the two constituent metals is normally one to one, i.e. of equal thickness. This comes about due to the way that the bimetallic strip is manufactured. The dominant method in the mass production of commercially available bimetallic strip involves either hot or cold rolling the two separate metals under intense pressures to produce interstitial bonding of the atoms at the bimetal interface Uhlig [5]. Under such conditions, it is expensive, because of setup costs, to make special separate metal thicknesses unless specifically required. Moreover, there is no data to support that different thicknesses of the bimetallic in the strip would have any performance benefit over equal thickness bimetal strip.

Cladding of metals Haga [6] is used to provide a product with a less expensive base metal that benefit from a thinner skin for decorative and or surface protection purposes, although clad metals are technically bimetallic metals, clad bimetallic strip is not being used for its functional bending qualities. Therefore the need to cater for separate material thicknesses is not required for most applications where the bimetallic bending qualities of a bimetallic strip are being exploited.

II. DERIVATION OF VARIOUS MATHEMATICAL MODELS

A. Timoshenko formula

From the Timoshenko [1], the radius of curvature of a bimetallic strip is given by:

\[
\rho = \frac{t_1}{6(\alpha_2 - \alpha_1)(T_h - T_c)(1 + m^2)} \left[ 3(1 + m^2) + (1 + m n) \left( m^2 + \frac{1}{m n} \right) \right]
\]

...............Eqn (1)

Where \(\rho\) is the radius of curvature, \(t\) total thickness of the strip, \(t_1\) and \(t_2\) are the individual material thicknesses. \(m = t_1/t_2\) is the ratio of thicknesses. \(n = E_2/E_1\) is the ratio of the Young’s Moduli. \(T_h\) and \(T_c\) are the hot and cold temperatures states. \(E_2, E_1\), is the linear Modulus of the two materials. \(\alpha_1\) & \(\alpha_2\) are the coefficients of linear thermal expansion for the two metals.
α₂ is assumed to be numerically larger than α₁.

Fig.1 shows a bimetallic strip in two states of heating, at state 1, at ambient temperature, the strip will be flat with no discernible radius of curvature R. At state 2, uniformly distributed heating will cause the strip to form into a radius of curvature.

Note that has a numerically higher coefficient of linear thermal expansion and thus naturally wants to extend further than the side with The differences, leads to internal stresses, forces and moments at the material interface, resulting in the bending as shown at state 2.

Figure 1 : Bimetallic strip in two states of heating

B. First order approximation model by Angel and Haritos

It is commonly known that the internal force developed within a metal bar by heating or cooling, can be written as follows:

\[ F = \alpha \Delta T AE \] \[ \text{Eqn (2)} \]

Where \( F \) is the force (N),
\( \alpha \) is the coefficient of linear expansion of the metal.
\( \Delta T \) is the temperature change of the metal from ambient (°K).
\( A \) is the cross-sectional area of the bar
\( E \) is the Young’s modulus of the material of the bar.

Eqn.(2) can be re-written in terms of the stress, \( \sigma = F/A \) since, thus the internal stress due to heating:

\[ \sigma = \alpha \Delta T E \] \[ \text{Eqn (3)} \]

By substitution of \( y = t/2 \) where \( y \) is assumed to be the distance from bimetal interface to the outer edge, this is also equal to half the total thickness of the bimetallic strip.

From the well-known simple bending equation

\[ \frac{\sigma}{E} = \frac{M}{I} \]

substituting and re-arranging using the first two terms of the simple bending equation, thus;

\[ R = \frac{t}{(\alpha_2 + \alpha_1) \Delta T} \] \[ \text{Eqn (4)} \]

Where:
\( R \) is the radius of curvature of the bimetallic strip to the bimetallic joint center line (mm).
\( t \) is the total thickness of the bimetallic strip(mm).

This is the first order estimate of the radius of curvature of the bimetallic strip with respect to temperature that approximates with the Timoshenko formula. It can be seen that the approximation of Eqn 4 tends to be more accurate as temperature increases and becomes more asymptotic to the Timoshenko curve.

C. Approximation model by correction factors (Angel and Haritos)

Referring to the paper by Angel and Haritos [9], further modification to the first order estimate (Eqn 4) is done by considering some modifiers and correction factors. For details, kindly refer [8], and the model is put here for completeness of the paper.

\[ R = \frac{t(\alpha_2 - \alpha_1)^2}{\Delta T(\alpha_2 + \alpha_1) \alpha_2^2} \] \[ \text{Eqn (5)} \]

where the correction factor is denoted by \( \beta = (\alpha_2 - \alpha_1)^2/(\alpha_2 + \alpha_1)\alpha_2^2 \)

D. Khathkate Singh Mirchandani (KSM) model

The authors of the paper have derived the model based on the first order approximation to the Timoshenko formula. Actually, the first order approximation does not incorporate the Young’s moduli of the two materials. However, the Eqn 4 represents the asymptotic nature of the Timoshenko curve. Also, the formula is simple to compute and does not need complex electronic devices. The authors wanted to check the variation in the same with respect to the Young’s moduli and hence have chosen the approximation as seen below.

\[ R = k t / (\alpha_2 + \alpha_1) \Delta T \] \[ \text{Eqn (6)} \]

Where:
\( R \) is the radius of curvature of the bimetallic strip to the bimetallic joint center line (m).
\( t \) is the total thickness of the bimetallic strip(m).
\( k \) = constant

\[ \text{If } E_1 < E_2 \quad n = E_1/E_2 \quad \text{else } n = E_2/E_1 \]

\[ R = t / (2n(\alpha_2 + \alpha_1) \Delta T) \] \[ \text{Eqn (6)} \]

Where:
\( R \) is the radius of curvature of the bimetallic strip to the bimetallic joint center line (mm).
\( t \) is the total thickness of the bimetallic strip(mm).

As seen in Eqn.6, by introducing a suitable function of the Young’s moduli in the Khathkate et. al model, the approximation is fairly improved as compared to other models in sections B and C.
III. SIMULATION DATA AND RESULTS

For the proof of the correlation between the new formula proposed in this paper, and Timoshenko’s original formula, two separate simulations were performed on the same dataset.

The first simulation was based around a bespoke Bimetallic strip SBC206-1 from Shivalik [7]; 100mm long x 5mm wide x 0.4mm thick, these were the starting values of the first simulation set.

\[ E_2 = 213 \text{ GPa} \]; Young’s modulus of Steel side of the bimetallic strip.

\[ E_1 = 145 \text{ GPa} \]; Young’s modulus of Invar 36 side of the bimetallic strip.

\[ \alpha_1 = 1.45 \times 10^{-6} \text{ m/m/°C} \]; coefficient of linear thermal expansion for Invar 36 side of the strip.

\[ \alpha_2 = 18.5 \times 10^{-6} \text{ m/m/°C} \]; coefficient of linear thermal expansion for Steel side of the strip.

\[ t = t_1 + t_2 \], both equal, total thickness of the strip.

\[ T_c = 20°C \] assumed ambient temperature constant throughout the simulation.

\[ T_h \] = Input variable temperature (°C).

\[ \Delta T = \text{change in temperature, (°C).} \]

\[ R = \text{radius of curvature evaluated by various models} \]

The figure below shows the performance of the various models to the Timoshenko formula. In order to compare the same, mean square error is computed for all the approximations. As seen in Figure, the approximation by Khatkhate et al shows the least error almost 2-3 times less than the other models.

![Figure 1: Comparison of performance of models (assuming constant expansion coefficient of Invar 36) with the Timoshenko formula.](image)

Figure 1: Comparison of performance of models (assuming constant expansion coefficient of Invar 36) with the Timoshenko formula.

However, we need to consider the fact that the coefficient of linear expansion of Invar 36 varies with temperature. The effect of temperature on the expansion coefficient has not been considered by other researchers in making their approximations. The effect of temperature on the expansion coefficient is seen in Table [9] below.

The effect of the same over a wider range of temperature can be seen in Figure 2 below. The model developed by Khatkhate et al shows slightly less error as compared to the other models. Thus the model does not work well as the coefficient of expansion increases with increase in temperature (see table above).

![Figure 2: Comparison of performance of models (with effect of temperature on expansion coefficient of Invar36) with the Timoshenko formula.](image)

Figure 2: Comparison of performance of models (with effect of temperature on expansion coefficient of Invar36) with the Timoshenko formula.

The model is very effective at low temperature up to 200 degree Celsius for low coefficient of thermal expansion of Invar 36 as seen in Figure below.

![Figure 3: Comparison of performance of models (over range of low expansion coefficient of Invar36) with the Timoshenko formula.](image)

Figure 3: Comparison of performance of models (over range of low expansion coefficient of Invar36) with the Timoshenko formula.
The mean square error of the model developed by Khatkhate et al is still 8-12 times less than the approximations developed by other researchers.

For simulation set 2, the following data was assumed:

The strip thickness \( t \), was varied from 0.4mm thick to 10mm thick. The two formulae of Eqn.1 and Eqn.6 were used to generate data values of the radius of curvature. The radii of curvature for simulation were plotted against the change of temperature for each thickness of bimetallic strip, see Figure 4. The results show that the correlation of the KSM model with the Timoshenko formula holds very good agreement for all thicknesses up to 10 mm at temperature below 180 degree C.

Figure 4 – Comparison of performance of KSM model with the Timoshenko formula over 0.4 to 10 mm.

IV MATLAB CODE FOR SIMULATION

MATLAB R2014a has been used as a simulation tool to generate the results in the paper. The MATLAB code can be made available for further improvement to anyone in the research community.

V CONCLUSIONS AND FUTURE WORK

As seen from Figure 1, we can conclude that the KSM model very well fits the Timoshenko formula. Also, the effect of the Young's moduli has been considered in making the model. The model is asymptotic in nature and easy to compute without need of electronic calculators.

<table>
<thead>
<tr>
<th>Model</th>
<th>Mean square error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Khatkhate et al</td>
<td>0.0109</td>
</tr>
<tr>
<td>1st order approx (Angel and Haritos)</td>
<td>0.1242</td>
</tr>
<tr>
<td>Correction factor (Angel and Haritos)</td>
<td>0.0816</td>
</tr>
</tbody>
</table>

Figure 2 takes into consideration the variation in expansion coefficient of Invar 36 with temperature which has not been considered by other researchers. The KSM model does not fit well in the higher temperature range.

Figure 3 shows that the KSM model fits well for the low coefficient of expansion of Invar 36 and can be very well used for calculating the radius of curvature. Figure 4 also shows that the KSM model works well with change in thickness over the range of temperature.

Furthermore, simulation with different materials of same thickness will be carried out in order to test the close agreement of the approximation with the Timoshenko formula. Further modification to the KSM model will be done to make it independent of the Young's moduli as suggested by other researchers. Also, suitable experimental setup will be fabricated to test the accuracy of the KSM model.

VI ACKNOWLEDGMENTS

The authors would like to express deepest appreciation towards Dr. Varsha Shah, Principal RCOE, Mumbai and Prof. Husain Jasdanwala whose invaluable guidance in this project has brought it to this stage.

Also, the first author would like to thank Professor Mirchandani for encouragement to teach the subject of Mechanical Measurements and Controls (MMC) which helped him in reading more about the work done by Timoshenko and other researchers on the Bimetallic Thermostats.

At last the authors express their sincere heartfelt gratitude to all the staff members of Mechanical Engineering Department who helped us directly or indirectly during this course of work.

REFERENCES