A new measure of Fuzzy Entropy and Fuzzy Divergence

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Abstract— **A new measure of Fuzzy Entropy and Fuzzy Divergence are obtained and particular case of four important and some other properties of the proposed measure.**

Keywords— ***Fuzzy Set, Fuzzy Entropy, Measure of fuzzy information.***

INTRODUCTION

The notion of fuzzy sets was proposed to tackling problems in which indefiniteness arising from a sort of intrinsic ambiguity plays a fundamental role. Fuzziness, a feature of uncertainty, results from the lack of sharp distinction of the boundary of a set, i.e., an individual is neither definitely a member of the set nor definitely not a member of it. new parametric generalized exponential entropy is proposed. This paper is organized as follows: some basic definition related to probability and fuzzy set theory are discussed. a new fuzzy entropy measure called, exponential fuzzy entropy of order-α is proposed and verifies the axiomatic requirements. some properties of the proposed measure are studied and limiting cases also discussed here and our conclusions are presented in

PRELIMINARIES

In this section we present some basic concepts related to probability theory and fuzzy sets which will be needed in the following analysis. First, let us cover probabilistic part of the preliminaries.

Let

$$\Delta_n = \left\{ P = (p_1, p_2, ...p_n) : p_i \geq 0.\ if \sum_{i=1}^{n} p_i = 1 \right\}, n \geq 2$$

be the set of n-complete probability distribution.

For any probability distribution $P = (p_1, p_2, ...p_n) \in \Delta_n$. Shannon entropy [11] is defined as

$$H(P) = -\sum_{i=1}^{n} p(x_i) \log p(x_i) \qquad (1)$$

Various generalized entropies have been introduced in the literature, taking the Shannon entropy as basic and have found applications in various disciplines such as economics, statistics, information processing and computing etc.

Generalizations of Shannon's entropy started with Renyi's` entropy [10] of order-α, given by

$$H_\alpha(P) = \frac{1}{(1-\alpha)} \log\left[\sum_{i=1}^{n} \left(p(x_i)\right)^\alpha \right], \alpha \neq 1, \alpha > 0 \qquad (2)$$

Pal and Pal [8, 9] analyzed the classical Shannon information entropy and proposed a information entropy called exponential entropy given by

$$E(P) = \sum_{i=1}^{n} p(x_i)\left(e^{(1-p(x_i))} - 1\right) \qquad (3)$$

These authors point out that, the exponential entropy has an advantage over Shannon's entropy. For the uniform probability distribution $P = \left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n} .... \frac{1}{n}\right)$ exponential entropy has a fixed upper bound

$$\lim_{n\to\infty} E\left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n} .... \frac{1}{n}\right) = (e-1) \qquad (4)$$

which is not the case for Shannon's entropy.

Corresponding to (2), Kvalseth [6] introduced and studied generalized exponential entropy of order-α, given by

$$E_\alpha(P) = \frac{\sum_{i=1}^{n} p(x_i)\left(e^{(1-p(x_i))} - 1\right)}{\alpha}, \alpha > 0 \qquad (5)$$

Definition 1: Let $X = (x_1, ...x_n)$ be a discrete universe of discourse. A fuzzy set A on X is characterized by a membership function $\mu_A(x)$: X → [0, 1].The value $\mu_A(x)$ of A at x ∈ X stands for the degree of membership of x in A.

Definition 2: A fuzzy set A^* is called a sharpened version of fuzzy set A if the following conditions are satisfied:

$$\mu_{A^*}(x) \leq \mu_A(x),\ \ if\ \mu_A(x) \leq 0.5; \forall i \quad and$$

$$\mu_{A^*}(x) \geq \mu_A(x),\ \ if\ \mu_A(x) \geq 0.5; \forall i$$

Definition 3: Let FS(X) denote the family of all FSs of universe X, assume $A, B \in FS(X)$ is given by $A = \{\langle x, \mu_A(x)\rangle | x \in X\}; B = \{\langle x, \mu_B(x)\rangle | x \in X\};$ then some set operations can be defined as follows:

i. $A^C = \{\langle x, 1-\mu_A(x), \mu_A(x)\rangle | x \in X\}$

ii. $A \cap B = \{\langle x, \min \mu_A(x), \mu_B(x)\rangle | x \in X\}$

iii. $A \cup B = \{\langle x, \max \mu_A(x), \mu_B(x)\rangle | x \in X\}$ First attempt to quantity the uncertainly associated with a fuzzy event in the context of discrete probabilistic frame work appears to have been made by Zadeh [14],who defined the entropy of fuzzy set A with respect to (X, P) as

$$H(A,P) = \sum_{i=1}^{n} \mu_A(x_i) p(x_i) \log p(x_i) \qquad (6)$$

De Luca and Termini [2] introduced a set of four axioms are widely accepted as criterion for defining any fuzzy entropy. In fuzzy set theory, the fuzzy entropy is a measure of fuzziness which expresses the amount of average ambiguity or difficulty in making a decision whether an element belongs to a set or not. A measure of fuzziness in a fuzzy set should have at least the following axioms:

PI (Sharpness): H(A) is minimum iff A is crisp set i.e. $\mu_A(x_i) = 0 \; or \; 1 \forall i.$

P2 (Maximality): H(A) is maximum iff A is most fuzzy set i.e. $\mu_A(x_i) = \frac{1}{2} \forall i.$

P3 (Resolution): $H(A^*) \le H(A)$ where A^* is a sharpened version of A.

P4 (Symmetry): $H(A) = H(A^C)$ where A^C is the complement set of A.

Since $\mu_A(x_i) \; and \; (1-\mu_A(x_i))$ gives same degree of fuzziness, therefore, De Luca and Termini [2] defined fuzzy entropy for a fuzzy set A corresponding to(1) as

$$H(A) = -\frac{1}{n}\sum_{i=1}^{n}\begin{bmatrix}\mu_A(x_i)\log \mu_A(x_i) + \\ (1-\mu_A(x_i))\log(1-\mu_A(x_i))\end{bmatrix} \qquad (7)$$

Later on Bhandari and pal [1] made a survey on information measures on fuzzy sets and gave some new measures of fuzzy entropy. Corresponding to (2) they have suggested the following measure:

$$H_\alpha(P) = \frac{1}{(1-\alpha)}\left[\sum_{i=1}^{n}\log\left[\mu_A^\alpha(x_i) + (1-\mu_A(x_i))^\alpha\right]\right], \quad \alpha \ne 1, \alpha > 0 \quad (8)$$

Pal and Pal [8,9] defined exponential fuzzy entropy for a fuzzy set corresponding (3) as

$$E(A) = \frac{1}{n(\sqrt{e}-1)}\sum_{i=1}^{n}\begin{bmatrix}\mu_A(x_i)e^{(1-\mu_A(x_i))} + \\ (1-\mu_A(x_i))e^{(1-\mu_A(x_i))} - 1\end{bmatrix} \qquad (9)$$

Throughout this paper, we denote the set of all fuzzy sets on X by FS(X).

In the next section we propose generalized fuzzy entropy measure corresponding to (4),called exponential fuzzy entropy of order-α and verify the axiomatic requirements.

EXPONENTIAL FUZZY ENTROPY OF ORDER

We proceed with the following formal definition:

Definition 4: Let A be the fuzzy set A fuzzy set defined on discrete universe of discourse $X = (x_1, \ldots\ldots x_n)$ having the membership values $\mu_A(x_i), i = 1,2\ldots\ldots n$. We define the exponential fuzzy entropy of order-α corre- sponding to (5), as

$$E_\alpha(A) = \frac{1}{n\left(e^{(1-0.5\alpha)}-1\right)}\sum_{i=1}^{n}\begin{bmatrix}\mu_A(x_i)e^{(1-\mu_A^\alpha(x_i))} + \\ (1-\mu_A(x_i)) \\ e^{(1-(1-\mu_A(x)^\alpha))}\end{bmatrix}, \alpha > 0 \quad (10)$$

Theorem:1 The measure (10) satisfy measure of fuzzy entropy.

Proof: Symmetry follows from the definition. We prove the properties (1) to (3) are satisfied by (10).

PI (Sharpness): First let $E_\alpha(A) = 0$, then

$$\left|\mu_A(x_i)e^{(1-\mu_A^\alpha(x_i))} + (1-\mu_A(x_i))e^{(1-(1-\mu_A(x)^\alpha))}\right| = 1 \qquad (11)$$

if $\alpha > 0$, (11) will hold when either $\mu_A(x_i) = 0$ or $\mu_A(x_i) = 1 \forall i = 1,2,3,\ldots..n.$

Conversely, if A is a crisp set , then either $\mu_A(x_i) = 0$ or $\mu_A(x_i) = 1 \forall i = 1,2,3,\ldots..n.$ it gives

$$\left|\mu_A(x_i)e^{(1-\mu_A^\alpha(x_i))} + (1-\mu_A(x_i))e^{(1-(1-\mu_A(x)^\alpha))}\right| = 1 \qquad (12)$$

$(ie).E_\alpha(A) = 0$. Hence $E_\alpha(A)$ iff A is crisp set.

P2 (Maximality): Let

$$E_\alpha(A) = \sum \int (\mu_A(x_i)) \qquad (13)$$

Where

$$\int(\mu_A(x_i)) = \frac{1}{n\left(e^{(1-0.5\alpha)}-1\right)}\sum_{i=1}^{n}\begin{bmatrix}\mu_A(x_i)e^{(1-\mu_A^\alpha(x_i))} + \\ (1-\mu_A(x_i))e^{(1-(1-\mu_A(x)^\alpha))}\end{bmatrix}, \alpha > 0 \quad (14)$$

Differentiating (14) w.r.t $\mu_A(x_i)$ we get,

$$\frac{\partial f(\mu_A(x_i))}{\partial \mu_A(x_i)} = \frac{1}{n\left(e^{(1-0.5\alpha)}-1\right)}\begin{bmatrix} e^{(1-\mu_A^\alpha(x_i))} - e^{(1-(1-\mu_A(x)^\alpha))} - \\ \alpha\begin{pmatrix}\mu_A^\alpha(x_i)e^{(1-\mu_A^\alpha(x_i))} - \\ (1-\mu_A(x_i))^\alpha e^{(1-(1-\mu_A(x)^\alpha))}\end{pmatrix}\end{bmatrix}, \quad (14)$$

Let $0 \le \mu_A(x_i) < 0.5,$ *then*

Thus $f(\mu_A(x_i))$ is a concave function which has a global maximum at $\mu_A(x_i) = 0.5.$ *Hence* $E_\alpha(A)$ is maximum iff A is the most fuzzy set.

$(i.e.) \; \mu_A(x_i) = 0.5. \forall i = 1,2,\ldots.n.$

P3 (Resolution): Since $H_\alpha(A)$ is increasing function of $\mu_A(x_i)$ in the range $[0,0.5)$ and the decreasing function of $\mu_A(x_i)$ in the range $(0,0.5]$ therefore,

$$\mu_{A^*}(x_i) \le \mu_A(x_i) \Rightarrow E_\alpha\left(A^*\right) \le E_\alpha(A) \quad in \quad [0,0.5) \quad \&$$

$$\mu_{A^*}(x_i) \ge \mu_A(x_i) \Rightarrow E_\alpha\left(A^*\right) \ge E_\alpha(A) \quad in \quad (0,0.5].$$

$$hence \ \ E_\alpha\left(A^*\right) \le E_\alpha(A).$$

P4 (Symmetry): It is clearly from the definition,

$$E_\alpha(A) = E_\alpha\left(A^C\right). \quad Hence \ \ prove the \ theorem.$$

IV. PROPERTIES OF EXPONENTIAL FUZZY ENTROPY OF ORDER-α

The measure of exponential fuzzy entropy of order-α has the following properties:

Theorem 2: For $A, B \in FS(X)$,

$$E_\alpha(A \cup B) + E_\alpha(A \cap B) = E_\alpha(A) + E_\alpha(B).$$

$$\Pr oof \ :Let$$

$$X_+ = \{x | x \in X, \mu_A(x_i) \ge \mu_B(x_i)\} \qquad (15)$$

$$X_- = \{x | x \in X, \mu_A(x_i) < \mu_B(x_i)\} \qquad (16)$$

$$where \ \mu_A(x) \ \& \ \mu_B(x) \ be \ the \ \ fuzzy membership$$

$$functions \ of \ \ A \& B respec.$$

$$E_\alpha(A \cup B) = \frac{1}{\left(e^{(1-0.5\alpha)} - 1\right)} \sum \left[\begin{array}{l} \mu_{A\cup B}(x_i) e^{\left(1-\mu^\alpha_{A\cup B}(x_i)\right)} + \\ \left(1-\mu_{A\cup B}(x_i)\right) e^{\left(1-\left(1-\mu_{A\cup B}(x_i)^\alpha\right)\right)-1} \end{array} \right] \qquad (17)$$

$$= \frac{1}{n\left(e^{(1-0.5\alpha)} - 1\right)} \left[\begin{array}{l} \sum_{X_I \in X_+} \left[\begin{array}{l} \mu_B(x_i) e^{\left(1-\mu^\alpha_B(x_i)\right)} + \\ \left(1-\mu_B(x_i)\right) e^{\left(1-\left(1-\mu_B(x_I)^\alpha\right)\right)} - 1 \end{array} \right] \\ + \sum_{X_I \in X_-} \left[\begin{array}{l} \mu_{AB}(x_i) e^{\left(1-\mu^\alpha_A(x_i)\right)} + \\ \left(1-\mu_A(x_i)\right) e^{\left(1-\left(1-\mu_A(x_I)^\alpha\right)\right)} - 1 \end{array} \right] \end{array} \right] \qquad (18)$$

Lemma:

$$For \ \ any \ \ A \in FS(X), and \ A^C of \ \ fuzzy set \ A.$$

$$E_\alpha(A) = E_\alpha\left(A^C\right) = E_\alpha\left(A \cup A^C\right) = E_\alpha\left(A \cap A^C\right).$$

V. CONCLUSIONS

This work introduces a Fuzzy Entropy and Fuzzy Divergence measure is exponential fuzzy entropy of order-α in the setting of fuzzy set theory.Some properties of this measure have been also studied.This measure generalizes Pal and Pal [9] exponential entropy and De-Luca and Termini [2] logarithmic entropy. Introduction of parameter-α provides new flexibility and wider application of exponential fuzzy entropy to different situations.

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