A New Application of Generalized Almost Increasing Sequence

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Abstract

A new result concerning absolute summability of infinite series using almost increasing sequence is obtained. An application gives some generaliztion of Sulaiman [3].

Keywords: Absolute summability, almost increasing sequence and sequence of bounded variation.

1 Introduction

Let $\sum a_n$ be an infinite series with sequence of partial sums (s_n). By $u_n^\alpha,\ t_n^\alpha$ we denote the n$^{\text{th}}$ Cesaro mean of order $\alpha > -1$ of the sequence $(s_n),\ (na_n)$ respectively, that is

$$u_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=0}^{n} A_{n-v}^{\alpha-1}s_v \tag{1.1}$$

$$t_n^\alpha = A_n^\alpha\sum_{v=0}^{n} A_{n-v}^{\alpha-1}va_v \tag{1.2}$$

The series $\sum a_n$ is summable $|C,\alpha|_k,\ k \geq 1$ if

$$\sum_{n=1}^{\infty} n^{k-1}|u_n^\alpha - u_{n-1}^\alpha|^k \equiv \sum_{n=1}^{\infty} n^{-1}|t_n^\alpha|^k < \infty. \tag{1.3}$$

For $\alpha = 1,\ |C,\alpha|_k$ summability reduces to $|C,1|_k$ summability.
Let (p_n) be a sequence of constants such that

$$P_n = p_0 + p_1 + ... + p_n \to \infty \text{ as } n \to \infty \tag{1.4}$$

The positive sequence (b_n) is said to be almost increasing sequence if there exists a positive increasing sequence (c_n) and two positive constants M and N

such that, $Mc_n \leq b_n \leq Nc_n$. Every increasing sequence is almost increasing sequence.

A sequence (λ_n) is said to be of bounded variation, denoted by $(\lambda_n) \in BV$ if

$$\sum_{n=1}^{\infty} |\Delta\lambda_n| = \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty.$$

2 Main Theorem

Here we generalized the Sulaiman theorem [3].

Theorem 2.1. Let $p > 0$, $p_n \geq 0$ and (p_n) be a non increasing sequence (Sulaiman [3]) (X_n) be almost increasing sequence if the following conditions (Bor [1]), (Mazhar [2]) and (Verma [4]). Where $\lambda_n \in BV$

$$\sum_{n=1}^{\infty} n|\Delta^2\lambda_n|X_n < \infty \tag{2.1}$$

$$|\lambda_n|X_n = O(1) \text{ as } n \to \infty \tag{2.2}$$

$$nX_n|\Delta\lambda_n| = 0(1) \text{ as } n \to \infty \tag{2.3}$$

$$\sum_{n=1}^{\infty} X_n|\Delta\lambda_n| < \infty \tag{2.4}$$

$$\psi_v = O(1) \text{ as } v \to \infty \tag{2.5}$$

$$v\Delta\psi_v = O(1) \text{ as } v \to \infty \tag{2.6}$$

and

$$\sum_{v=1}^{n} \frac{v^{\delta k-1}}{X_v^{k-1}}|t_v|^k = O(X_n) \text{ as } n \to \infty \tag{2.7}$$

are satisfied, then the series $\sum a_n\lambda_n\psi_n$ is summable $|C, 1, \delta|_k$, $k \geq 1$, $\delta \geq 0$.

Proof. Let T_n be the n-th $(C, 1)$ means of the sequence $(na_n\lambda_n\psi_n)$.

Therefore

$$T_n = \frac{1}{n+1}\sum_{v=1}^{n} va_v\lambda_v\psi_v.$$

Abel's transformation gives

$$T_n = \frac{1}{n+1}\left(\sum_{v=1}^{n-1} \Delta(\lambda_v\psi_v)\sum_{r=1}^{v} r\ a_r + \lambda_n\psi_n\sum_{v=1}^{n} v\ a_v\right)$$

$$= \frac{1}{n+1}\left(\sum_{v=1}^{n-1}(v+1)t_v\ \Delta\psi_v\lambda_v + \sum_{v=1}^{n-1}(v+1)t_v\psi_{v+1}\Delta\lambda_v\right) + t_n\psi_n\lambda_n.$$

$$= T_{n,1} + T_{n,2} + T_{n,3}.$$

In order to complete the proof, by Minkowaski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{\delta k-1}|T_{n,j}|^k < \infty,\ j = 1, 2, 3$$

Applying Hölder inequality, we have

$$\sum_{n=2}^{m+1} n^{\delta k-1}|T_{n,1}|^k = \sum_{n=2}^{m+1} n^{\delta k-1}\left|\frac{1}{n+1}\sum_{v=1}^{n-1}(v+1)t_v\Delta\psi_v\lambda_v\right|^k$$

$$= O(1)\sum_{n=2}^{m+1}\frac{n^{\delta k-1}}{n^k}\sum_{v=1}^{n-1} v^k|t_v|^k|\Delta\psi_v|^k|\lambda_v|^k \cdot \left(\sum_{v=1}^{n-1} 1\right)^{k-1}$$

$$= O(1)\sum_{n=2}^{m+1}\frac{n^{\delta k-1}}{n^k}\sum_{v=1}^{n-1} v^k|t_v|^k|\Delta\psi_v|^k|\lambda_v|^k(n)^{k-1}$$

$$= O(1)\sum_{n=2}^{m+1} n^{\delta k-2}\sum_{v=1}^{n-1} v^k|t_v|^k|\Delta\psi_v|^k|\lambda_v|^k$$

$$= O(1)\sum_{v=1}^{m} v^k|t_v|^k|\Delta\psi_v|^k|\lambda_v|^k\sum_{n=v+1}^{m+1} n^{\delta k-2}$$

$$= O(1)\sum_{v=1}^{m} v^k|t_v|^k|\Delta\psi_v|^k|\lambda_v|^k\int_v^\infty x^{\delta k-2}dx$$

$$= O(1)\sum_{v=1}^{m} v^k|t_v|^k|\Delta\psi_v|^k|\lambda_v|^k(v)^{\delta k-1}$$

$$= O(1)\sum_{v=1}^{m} v^{\delta k-1}|t_v|^k|v\Delta\psi_v|^k|\lambda_v|^k$$

$$= O(1)\sum_{v=1}^{m} v^{\delta k-1}|t_v|^k|\lambda_v|^k$$

$$= O(1)\sum_{v=1}^{m}\frac{v^{\delta k-1}|t_v|^k|\lambda_v|}{X_v^{k-1}}$$

$$= O(1)\sum_{v=1}^{m}|\Delta\lambda_v|\sum_{r=1}^{v}\frac{|t_r|^k\ r^{\delta k-1}}{X_v^{k-1}} + O(1)|\lambda_m|\sum_{v=1}^{m}\frac{|t_v|^k v^{\delta k-1}}{X_v^{k-1}}$$

$$= O(1)\sum_{v=1}^{m-1} X_v|\Delta\lambda_v| + O(1)X_m|\lambda_m| = O(1)$$

$$
\begin{aligned}
\sum_{n=2}^{m+1} n^{\delta k-1}|T_{n,2}|^k &= \sum_{n=2}^{m+1} n^{\delta k-1}\left|\frac{1}{n+1}(v+1)t_v\psi_{v+1}\Delta\lambda_v\right|^k \\
&= O(1)\sum_{n=2}^{m+1}\frac{n^{\delta k-1}}{n^k}\sum_{v=1}^{n-1} v^k|t_v|^k|\psi_{v+1}|^k|\Delta\lambda_v|^k \\
&= O(1)\sum_{n=2}^{m+1}\frac{n^{\delta k-1}}{n^k}\sum_{v=1}^{n-1}\frac{v^k|t_v|^k|\psi_{v+1}|^k|\Delta\lambda_v|}{X_v^{k-1}}\cdot\left(\sum_{v=1}^{n-1}X_v|\Delta\lambda_v|\right)^{k-1} \\
&= O(1)\sum_{v=1}^{m}\frac{v^k|tv|^k|\Delta\lambda_v|}{X_v^{k-1}}\sum_{n=v+1}^{m+1} n^{\delta k-k-1} \\
&= O(1)\sum_{v=1}^{m}\frac{v^k|t_v|^k|\Delta\lambda_v|}{X_v^{k-1}}\cdot v^{\delta k-k} \\
&= O(1)\sum_{v=1}^{m}\frac{|t_v|^k v^{\delta k-1}}{X_v^{k-1}}|v\Delta\lambda_v| \\
&= O(1)\sum_{v=1}^{m}|\Delta(v|\Delta\lambda_v|)|\sum_{r=1}^{v}\frac{|t_r|^k r^{\delta k-1}}{X_r^{k-1}} + O(1)|\lambda_m\sum_{v=1}^{m}\frac{|t_v|^k\ v^{\delta k-1}}{X_v^{k-1}} \\
&= O(1)\sum_{v=1}^{m-1} v|\Delta^2\lambda_V|X_v + O(1)\sum_{v=1}^{m-1}X_v|\Delta\lambda_v| + O(1)m|\Delta\lambda_m|X_m \\
&= O(1)
\end{aligned}
$$

And

$$
\begin{aligned}
\sum_{n=1}^{m} n^{\delta k-1}|T_{n,3}|^k &= \sum_{n=1}^{m} n^{\delta k-1}|t_n\psi_n\lambda_n|^k \\
&= O(1)\sum_{n=1}^{m}\frac{|t_n|^k\cdot n^{\delta k-1}}{X_n^{k-1}}|\lambda_n| \\
&= O(1) \text{ as in the case of } T_{n,1}
\end{aligned}
$$

This completes the proof of theorem. □

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