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A Mathematical Model Of Hiv/Aids Dynamics With Four Compartments

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Abstract

The research work considers four compartmentalized mathematical model of HIV/AIDS disease dynamics of the susceptible, removed, latent and infected classes which are age structure free. The susceptibles are virus free but are prone to infection through interaction with the latent and infected classes. Members of the removed class are not prone to infection due to their adherence to warnings or change in behavior through public enlightenment. The latent class refers to members of the population that contracted the HIV virus but have no symptom of AIDS. Members of the infected class are already manifested with AIDS. The partitioning of the population resulted into a set of four ordinary differential equations. Parameter values were used to represent the consequential interactive characteristics of the population. The equilibrium states and the corresponding characteristics equation were obtained to help in the analysis of the model. The Bellman and Cooke's theorem is applied to analysed the non zero equilibrium state for stability or otherwise and bounds obtained for sustenance of the population.

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1. Introduction

The research work proposes a deterministic mathematical model which is a system of Ordinary Differential Equations(ODE).ODE form very important mathematical tools used in producing models of physical and biological processes. Burghes and Wood [3] opines that "…it could even be claimed that the spread of modern industrial civilization, for better or for worse, is partly a result of man's ability to solve the differential equations which govern so many of our industrial processes, be them chemical or engineering." According to Benyah [2], mathematical modeling is an evolving process, as new insight is gained the process begins again as additional factors are considered.

The population is partitioned into four compartments of the susceptibles S(t), this is the class in which members are virus free but are prone to infection by interaction with the latent and the infected classes; the second class is the removed R(t), which is the class of those not susceptible to infection, possibly due to their yielding to warnings or changed behavior as a result of public awareness campaign or enlightenment; the third class is the latent L(t), this is the class of those that have contracted the virus, but have no symptom of the AIDS disease, the members of this class are still active in the population both sexually and economically. The last class is the infected I(t), in this class, members already have the manifestation of the symptoms of AIDS; this class is generally weak and inactive.

It is assumed that while the new birth of S(t) and R(t) are born into S(t), the off-springs of L(t) are divided between S(t) and L(t) in the proportion θ and 1 - θ respectively, that is, a proportion 1 - θ of the off-springs of L(t) are born with the virus.The four classes have a natural death rate of μ, while the infected class L(t) has additional death modulus δ arising from the weight of infection.Members of the class S(t) move into the class R(t) due to change in behavior or/and as a result of effective public campaign at a rate γ and members of the class L(t) move in to the class I(t) at the rate τ. Members of the class S(t) moved in to the class L(t) at the rate α by interacting with L(t) or/and I(t)

2. The model equations

S = Susceptible class
R = Removed class
L = Latent class
I = Infected class
P = population

P = S + R + L + I

$$\frac{ds}{dt} = (\beta - \mu - \Upsilon\gamma)S(t) + \beta\mathcal{R}(t) + \theta\beta L(t) - \propto s(t)[L(t) + I(t)] \quad (1)$$

$\frac{dR}{dt} = [ɤɤ\gamma s(t) - \mu R(t)]$ (2)

$\frac{dL}{dt} = [(1-0)\beta - \mu - \tau]L(t) + \propto S(t)[L(t) + I(t)]$ (3)

$\frac{dI}{dt} = (\beta - \mu - \delta)I(t) + \tau L(t)$ (4)

With the parameters given by

β = Natural birth rate for the population

μ = Natural death rate for the infection

δ = Death modulus due to infection

$\propto$ = Rate of contracting the HIV virus

$ɤɤ\gamma$ = Rate of removal of the susceptibles into the removed class due to public campaign

τ = Rate of flow from the latent class into the infected class

θ = The proportion of the off-springs of the latent which are virus free at birth $0 \leq \theta \leq 1$

t = Time

3. Equilibrum state of the model

At equilibrium state let

(S (t), R (t), L(t), I (t)) = (w, x, y, z) (4)

Then, we have that

$(\beta - \mu - ɤ\gamma ɤ)$ w + $\beta x + \theta\, \beta y - \propto w[y + z] = 0$ (5)

$ɤɤ\gamma w - \mu x = 0$ (6)

$[(1-\theta)\,\beta - \mu - \tau\,]$ y+$\propto$ w [y + z] = 0 (7)

$(\beta - \mu - \delta\,)Z + \tau\, y = 0$ (8)

From (8)

$(\beta - \mu - \delta\,)\, z = -\tau\, y$

$z = \frac{\tau y}{\mu+\delta-\beta}$ (9)

Substituting (9) in to (7), we obtained

$[(1-\theta)\beta - \mu - \tau]\, y + \propto w \left[y + \frac{\tau y}{\mu+\delta-\beta}\right] = 0$

$\{[(1-ɤ\theta)\beta - \mu - \tau] + \propto w \left[1 + \frac{\tau}{\mu+\delta-\beta}\right]\}\, y = 0$

$y = 0$ (10)

$0r$

$(\mu + \delta - \beta)[(1-ɤ\theta)\beta - \mu - \tau] + \propto w(\mu + \delta - \beta + \tau) = 0$

$w = \frac{(\mu+\delta-\beta)[(1-ɤɤ\theta)\beta-\mu-\tau]}{\propto(\beta-\mu-\delta-ɤ\gamma ɤ)}$ (11)

Adding equations (5) *and* (7), *we obtain*

$(\beta - \mu - ɤ\gamma ɤ)$ w + $\beta x + \theta\, \beta y - \propto w[y + z] + \beta y - \theta\, \beta y - \mu y - \tau y + \propto w[y + z] = 0$

$(\beta - \mu - ɤ\gamma)$ w+ $\beta\, x + \beta y - \mu y - \tau y = 0$

$(\beta - \mu - \tau)\, y = -[(\beta - \mu - ɤ\gamma)w + \beta\, x]$ (12)

$y = \frac{(\beta-\mu-ɤɤ\gamma)w+\beta x}{\mu+\tau-\beta}$ (13)

Substituting y = 0 into (12)

$[(\beta - \mu - ɤ\gamma)$ w+ $\beta\, x = 0$ (14)

From equation (6)

$ɤ\gamma$ w – $\mu\, x = 0$

$w = \frac{\mu}{\gamma} x$ (15)

Substituting (15) into (14), we obtain

$\{[(\beta - \mu - ɤ\gamma)\, {}^{\mu}/_{ɤ}\gamma]\, x + \beta\, x\,\} = 0$

$x = 0$

Substituting y = 0 into (8) gives

z = 0

and substituting x = y = z = 0 into (5), we obtain w =0

Hence, the zero equilibrium state is given by

(w, x,y,z) = (0,0,0,0) (16)

From (6)

$\mu x = ɤ\gamma$ w

$x = \frac{\gamma ɤ w}{\mu}$ (17)

Substituting (11) into (17), we obtain

$x = \frac{\gamma ɤ(\mu+\delta-\beta)[(1-\theta)\beta-\mu-\tau]}{\propto(\beta-\mu-\delta-\gamma ɤ)}$ (18)

Putting (11) and (18) into (13) gives

$y = \frac{[\mu(\beta-\mu-\gamma ɤɤ)+\beta ɤ\gamma](\mu+\delta-\beta)[(1-\theta)\beta-\mu-\tau]}{\propto\mu(\beta-\mu-\delta-\gamma ɤɤ)(\mu+\tau-\beta)}$ (19)

Substituting (19) into (9) we obtain

$z = \{\frac{[\mu(\beta-\mu-\gamma ɤ)+\beta\gamma ɤ](\mu+\delta-\beta)[(1-\theta)\beta-\mu-\tau]}{\propto\mu(\beta-\mu-\delta-\gamma ɤ)(\mu+\tau-\beta)}\}\frac{1}{\mu+\delta-\beta}$

4. The characteristics equation

The Jacobian determinant for the system with the eigen value λ is given by

$$\begin{vmatrix} \beta - \mu - \gamma ɤ - \propto y - \propto Z - \lambda & \beta & \theta\,\beta - \propto & -\propto w \\ \gamma ɤ & \mu - \lambda & 0 & \propto w \\ \propto y + \propto Z & 0 & [(1-\theta)\beta - \mu - \tau + \propto w - \lambda & \\ 0 & 0 & \tau & \beta - \mu - \tau - \lambda \end{vmatrix}$$

$=0$

$[\beta - \mu - \gamma\upsilon - \propto y - \propto Z - \lambda](-\mu - \lambda)\{[(1-\theta)\beta - \mu - \tau + \propto w - \lambda][\beta - \mu - \delta - \lambda] - [\propto \tau w]\}$

$-\beta\gamma\upsilon\{[(1-\theta)\beta - \mu - \tau + \propto w - \lambda][\beta - \mu - \delta - \lambda] - [\propto \tau w]\}$

$+(\theta\beta - \propto w)(\mu + \lambda)(\propto y + \propto Z)(\beta - w - \delta - \lambda) + \propto w(\mu + \lambda)(\propto y + \propto Z)\tau = 0$

Hence, the characteristics equation is given by

$\{[\beta - \mu - \gamma\upsilon - \propto y - \propto Z - \lambda](-\mu - \lambda) - \beta\gamma\upsilon\}\{[(1-\theta)\beta - \mu - \tau + \propto w - \lambda][\beta - \mu - \delta - \lambda]$

$-[\propto \tau w]\} + \propto (y + Z)(\mu + \lambda)\{(\theta\beta - \propto w)(\beta - \mu - \delta - \lambda) + \propto \tau w\} = 0 \quad (21)$

5. Stability analysis of the zero equilibrium state

At the zero equilibrium state
$(w, x, y, z) = (0,0,0,0)$
The characteristics equation takes the form
$\{[\beta - \mu - \upsilon\gamma\upsilon - \lambda](-\mu - \lambda) - \beta\gamma\upsilon\}\{[(1-\theta)\beta - \mu - \tau - \lambda][\beta - \mu - \delta - \lambda]\} = 0 \quad (22)$

If $(\beta - \mu - \upsilon\gamma\upsilon - \lambda)(-\mu - \lambda) - \beta\upsilon\gamma = 0$

We have

$-\beta\mu + \mu^2 + \gamma\upsilon\mu + \lambda\mu - \beta\lambda + \mu\lambda + \upsilon\gamma\lambda + \lambda^2 - \beta\upsilon\gamma = 0$

$\lambda^2 + 2\lambda\mu - \lambda\beta + \upsilon\gamma\lambda + \mu^2 - \beta\mu + \gamma\upsilon\mu - \beta\gamma\upsilon = 0$

$\lambda^2 + (2\mu - \beta + \gamma\upsilon)\lambda - \beta\mu - \beta\gamma\upsilon + \mu^2 + \gamma\upsilon\mu = 0$

$\lambda^2 + (2\mu - \beta + \upsilon\gamma)\lambda - \beta(\mu - \gamma\upsilon) + \mu(\mu\upsilon + \gamma) = 0$

$\lambda^2 + (2\mu - \beta + \gamma\upsilon)\lambda + [(\mu + \gamma\upsilon)(\mu - \beta)] = 0$

Which is a quadratic equation in λ

Hence $\lambda_1 = \beta - \mu \quad (23)$

$\lambda_2 = -\mu - \gamma\upsilon \quad (24)$

Similarly, from equation (22),

$\lambda_3 = (1-\theta)\beta - \mu - \tau \quad (25)$

$\lambda_4 = \beta - \mu - \delta \quad (26)$

We note that $\lambda_2 < 0$, we have that the system will be stable at the origin if $\beta < \upsilon\mu$, that is, when the death modulus is higher than the birth modulus.

6. Stability analysis of the non-zero equilibrium state

For the non zero equilibrium state,

w = $\frac{(\mu+\delta-\beta)[(1-\theta)\beta-\mu-\tau]}{\propto(\beta-\mu-\delta-\gamma\upsilon)}$,

$x = \frac{\gamma\upsilon(\mu+\delta-\beta)[(1-\theta)\beta-\mu-\tau]}{\propto\mu(\beta-\mu-\delta-\gamma\upsilon)}$,

y = $\frac{[\mu(\beta-\mu-\gamma\upsilon)+\beta\gamma](\mu+\delta-\beta)[(1-\theta)\beta-\mu-\tau]}{\propto\mu(\beta-\mu-\delta-\gamma\upsilon)(\mu+\tau-\beta)}$,

z = $\frac{[\mu(\beta-\mu-\gamma\upsilon)+\beta\gamma\upsilon][(1-\theta)\beta-\mu-\tau]}{\propto\mu(\beta-\mu-\delta-\gamma\upsilon)(\mu+\tau-\beta)}$

To analyse the non zero state for stability, we shall apply the Bellman and Cooke's theorem [1] to the characteristics equation (21) in the form H (λ) = 0

H (λ) = $\{[\beta - \mu - \upsilon\gamma\upsilon - \propto y - \propto z - \lambda](-\mu - \lambda) - \beta\upsilon\gamma\}\{[(1-\theta)\beta - \mu - \tau + \propto w - \lambda][\beta - \mu - \delta - \lambda]$

$-[\propto \tau w]\} + \propto (y + z)(\mu + \lambda)\{(\theta\beta - \propto w)(\beta - \mu - \delta - \lambda) + \propto \tau w\}$

H (λ) = $\{-\mu(\beta - \mu - \gamma - \propto y - \propto z - \lambda) - \lambda(\beta - \mu - \gamma\upsilon\upsilon - \propto y - \propto z - \lambda) - \beta\gamma\upsilon\}$

$\{[(1-\theta)\beta - \mu - \tau + \propto w - \lambda](\beta - \mu - \delta) - \lambda[(1-\theta)\beta - \mu - \tau + \propto w - \lambda] - [\propto \tau w]\}$

+ {∝(y + z) $(\mu + \lambda)(\theta\beta - \propto w)(\beta - \mu - \delta - \lambda) + \propto^2 \tau w$ (y + z) $(\mu + \lambda)\}$

H (λ) = $\{-\mu(\beta - \mu - \upsilon\gamma\upsilon - \propto y - \propto z) + \lambda\mu - \lambda(\beta - \mu - \upsilon\gamma\upsilon - \propto y - \propto z)\lambda^2 - \beta\gamma\upsilon\}$

$\{[(1-\theta)\beta - \mu - \tau + \propto w](\beta - \mu - \delta) - \lambda(\beta - \mu - \delta) - \lambda[(1-\theta)\beta - \mu - \tau + \propto w] + \lambda^2 - \propto \tau w\}$

$+ \{\propto(y+z)(\mu+\lambda)(\theta\beta-\propto w)(\beta-\mu-\delta) - \lambda\propto(y+z)(\mu+\lambda)(\theta\beta-\propto w) + \propto^2 \tau w(y+z)(\mu+\lambda)\}$

$H(\lambda) = \{-\mu(\beta-\mu-\gamma\gamma\gamma-\propto y-\propto z) + \lambda\mu - \lambda(\beta-\mu-\gamma\gamma\gamma-\propto y-\propto z) + \lambda^2 - \beta\gamma\gamma\}$

$\{[(1-\theta)\beta-\mu-\tau+\propto w](\beta-\mu-\delta) - \lambda(\beta-\mu-\delta) - \lambda[(1-\theta)\beta-\mu-\tau+\propto w] + \lambda^2 - \propto\tau w\}$

$+\{\propto\mu(y+z)(\theta\beta-\propto w)(\beta-\mu-\delta) + \lambda\propto(y+z)(\theta\beta-\propto w)(\beta-\mu-\delta) - \lambda\mu\propto$

$(y+z)(\theta\beta-\propto w) - \lambda^2\propto(y+z)(\theta\beta-\propto w) + \propto^2\mu\tau\gamma w(y+z) + \lambda\propto^2\tau\gamma w(y+z)\}$

$H(\lambda) = \{\lambda^2 + \lambda[\mu - (\beta-\mu-\gamma\gamma\gamma-\propto y-\propto z)] - [\mu(\beta-\mu-\gamma\gamma\gamma-\propto y-\propto z)] + \beta\gamma\gamma]\}\{\lambda^2 - \lambda$

$[(\beta-\mu-\delta) + ((1-\theta)\beta-\mu-\tau+\propto w)] + [((1-\theta)\beta-\mu-\tau+\propto w)(\beta-\mu-\delta) - \propto\tau w\} + \{-$

$\lambda^2\propto(y+z)(\theta\beta-\propto w) + \lambda[\propto(y+z)(\theta\beta-\propto w)(\beta-\mu-\delta) - \mu\propto(y+z)(\theta\beta-\propto w) + \propto^2\boldsymbol{\tau}w(y+z)] + \propto\mu(y+z)(\theta\beta-\propto w)(\beta-\mu-\delta) + \propto\mu^2\boldsymbol{\tau}w(y+z)\}$

$H(\lambda) = \lambda^2\{\lambda^2 - \lambda[(\beta-\mu-\delta) + ([(1-\theta)\beta-\mu-\tau+\propto w)] + [((1-\theta)\beta-\mu-\tau+\propto w)(\beta-\mu-\delta)$

$-\propto\boldsymbol{\tau}w]\} + \lambda[\mu - (\beta-\mu-\gamma\gamma\gamma-\propto y-\propto z)]\{\lambda^2 - \lambda[(\beta-\mu-\delta) + ((1-\theta)\beta-\mu-\tau+\propto$

$w)] + [((1-\theta)\beta-\mu-\tau+\propto w)(\beta-\mu-\delta) - \propto\boldsymbol{\tau}w]\} - [\mu(\beta-\mu-\gamma\gamma\gamma-\propto y-\propto z) + \beta\gamma\gamma]$

$\{\lambda^2 - \lambda[(\beta-\mu-\delta) + ((1-\theta)\beta-\mu-\tau+\propto w)] + [((1-\theta)\beta-\mu-\tau+\propto$

$w)(\beta-\mu-\delta) - \propto\boldsymbol{\tau}w]\} + \{-\lambda^2\propto(y+Z)(\theta\beta-\propto w) + \lambda[\propto(y+Z)$

$(\theta\beta-\propto w)(\beta-\mu-\delta) - \mu\propto(y+Z)(\theta\beta-\propto w) + \propto^2\tau w(y+Z)] + \propto\mu(y+Z)(\theta\beta-\propto w)(\beta-\mu-\delta) + \propto^2\mu\tau w(y+Z)\}$

$H(\lambda) = \lambda^4 - \lambda^3[(\beta-\mu-\delta)] + ((1-\theta)\beta-\mu-\tau+\propto w)] + \lambda^2[((1-\theta)\beta-\mu-\tau+\propto w)(\beta-\mu-\delta) - \propto\tau w]$

$+\lambda^3[\mu - (\beta-\mu-\gamma\gamma\gamma-\propto y-\propto z)] - \lambda^2[\mu - (\beta-\mu-\gamma\gamma\gamma-\propto y-\propto z)]$

$[(\beta-\mu-\delta) + [(1-\theta)\beta-\mu-\tau+\propto w)] + \lambda[\mu - (\beta-\mu-\gamma\gamma\gamma-\propto y-\propto z)][((1-\theta)\beta-\mu-\tau+\propto w)(\beta-\mu-\delta) - \propto\tau w] - \lambda^2[\mu(\beta-\mu-\gamma\gamma\gamma-\propto y-\propto z + \beta\gamma\gamma] + \lambda[\mu(\beta-\mu-\gamma\gamma\gamma-\propto y-\propto z) + \beta\gamma\gamma][(\beta-\mu-\delta) + [(1-\theta)\beta-\mu-\tau+\propto w)] - [\mu(\beta-\mu-\gamma\gamma\gamma-\propto y-\propto z) + \beta\gamma\gamma]$

$[((1-\theta)\beta-\mu-\tau+\propto w)(\beta-\mu-\delta) - \propto\tau w] - \lambda^2\propto(y+Z)(\theta\beta-\propto w) + \lambda[\propto(y+Z)(\theta\beta-\propto w)$

$(\beta-\mu-\delta) - \mu\propto(y+Z)(\theta\beta-\propto w) + \propto^2\tau w(y+Z)] + \propto\mu(y+Z)(\theta\beta-\propto w)$

$(\beta-\mu-\delta) + \propto^2\mu\tau w(y+z)$

$H(\lambda) = \lambda^4 - \lambda^3\{[\mu - (\beta-\mu-\gamma\gamma\gamma-\propto y-\propto z) - (\beta-\mu-\delta) + ((1-\theta)\beta-\mu-\tau+\propto$

$w)\} + \lambda^2\{[(1-\theta)\beta-\mu-\tau+\propto w)(\beta-\mu-\delta) - \propto\tau w] - [\mu - (\beta-\mu-\gamma\gamma\gamma-\propto y-\propto z)][(\beta-\mu-\delta)] +$

$((1-\theta)\beta-\mu-\tau+\propto w)] - [\mu(\beta-\mu-\gamma\gamma\gamma-\propto y-\propto z + \beta\gamma\gamma] - \propto(y+Z)(\theta\beta-\propto w)\} + \lambda\{[\mu -$

$(\beta-\mu-\gamma\gamma\gamma-\propto y-\propto z)][((1-\theta)\beta-\mu-\tau+\propto w)(\beta-\mu-\delta) - \propto\tau w] + [\mu(\beta-\mu-\gamma\gamma\gamma-\propto y-\propto z) + \beta\gamma\gamma][(\beta-\mu-\delta) + ((1-\theta)\beta-\mu-\tau+\propto w)] + [\propto(y+z)(\theta\beta-\alpha w)(\beta-\mu-\delta) - \mu\propto(y+z)$

$(\theta\beta-\propto w) + \propto^2\tau w(y+Z)\} + \propto\mu(y+Z)(\theta\beta-\propto w)(\beta-\mu-\delta) + \propto^2\mu\tau w(y+Z) - [\mu(\beta-\mu-\gamma\gamma\gamma-\propto y-\propto z) + \beta\gamma\gamma][((1-\theta)\beta-\mu-\tau+\propto w)(\beta-\mu-\delta) - \propto\tau w]$

We now set $\lambda = ip$,

$H(ip) = (ip)^4 + (ip)^3\{[\mu - (\beta-\mu-\gamma\gamma-\propto y-\propto z)] - [(\beta-\mu-\delta) + ((1-\theta)\beta-\mu-\tau+\propto w)\} + (ip)^2\{[((1-\theta)\beta-\mu-\tau+\propto w)(\beta-\mu-\delta) - \propto\tau w] - [\mu(\beta-\mu-\gamma\gamma-\propto y$

$-\propto z)][(\beta-\mu-\delta) + ((1-\theta)\beta-\mu-\tau+\propto w)] - [\mu(\beta-\mu-\gamma\gamma-\propto y-\propto z) + \beta\gamma\gamma\gamma]$

$-\propto(y+z)(\theta\beta-\propto w)\} + ip\{[\mu - (\beta-\mu-\gamma\gamma\gamma-\propto y-\propto z)][((1-\theta)\beta-\mu-\tau+\propto w)(\beta-\mu-\delta) - \propto\tau w] + [\mu(\beta-\mu-\gamma\gamma-\propto y-$

$\propto z)+\beta\gamma\Upsilon\Upsilon]\ [(\beta-\mu-\delta)+((1-\theta)\beta-\mu-\tau+\propto w)]\ [\propto(y+z)(\theta\beta-\propto w)(\beta-\mu-\delta)-\mu\propto(y+z)(\theta\beta-\propto w)+\propto^2\tau w(y+z)]\}+\propto\mu(y+z)(\theta\beta-\propto w)(\beta-\mu-\delta)+\propto^2\mu\tau w(y+z)-(\mu(\beta-\mu-\gamma-\propto y-\propto z)+\beta\Upsilon\gamma\Upsilon)\ [((1-\theta)\beta-\mu-\tau+\propto w)(\beta-\mu-\delta)-\propto\tau w]$

$H(\imath p)=p^4-\imath p^3\{[(\mu-(\beta-\mu-\gamma\Upsilon-\propto y-\propto z)]-[(\beta-\mu-\delta)+((1-\theta)\beta-\mu-\tau+\propto w)]\}$

$-p^2\{[(1-\theta)\beta-\mu-\tau+\propto w)(\beta-\mu-\delta)-\propto\tau w]-[\mu(\beta-\mu-\gamma\Upsilon-\propto y-\propto z)][(\beta-\mu-\delta)+((1-\theta)\beta-\mu-\tau+\propto w)]-[\mu(\beta-\mu-\gamma\Upsilon-\propto y-\propto z)+\beta\gamma\Upsilon]-\propto(y+z)(\theta\beta-\propto w)\}+\imath p\{[\mu-(\beta-\mu-\Upsilon\gamma\Upsilon-\propto y-\propto z)]\ [((1-\theta)\beta-\mu-\tau+\propto w)(\beta-\mu-\delta)-\propto\tau w]+[\mu(\beta-\mu-\gamma\Upsilon-\propto y-\propto z)+\beta\Upsilon]\ [(\beta-\mu-\delta)+((1-\theta)\beta-\mu-\tau+\propto w)][\propto(y+z)(\theta\beta-\propto w)(\beta-\mu-\delta)-\mu\propto(y+z)(\theta\beta-\propto w)+\propto^2\tau w(y+z)]\}\mp\propto\mu(y+z)(\theta\beta-\propto w)(\beta-\mu-\delta)+\propto^2\mu\tau w(y+z)-(\mu(\beta-\mu-\gamma\Upsilon-\propto y-\propto z)+\beta\Upsilon]\ [((1-\theta)\beta-\mu-\tau+\propto w)(\beta-\mu-\delta)-\propto\tau w]$

Resolving into real and imaginary parts,

$H(\imath p)=F(p)+\imath G(p).$

F(p) and G(p) are given respectively by

$F(p)=p^4-p^2\{[((1-\theta)\beta-\mu-\tau+\propto w)(\beta-\mu-\delta)-\propto\tau w]-[\mu(\beta-\mu-\gamma\Upsilon-\propto y-\propto z)][(\beta-\mu-\delta)+((1-\theta)\beta-\mu-\tau+\propto w)]-[\mu(\beta-\mu-\gamma\Upsilon-\propto y-\propto z)+\beta\gamma\Upsilon]-\propto(y+z)(\theta\beta-\propto w)\}+\propto\mu(y+z)(\theta\beta-\propto w)(\beta-\mu-\delta)+\propto^2\mu\tau w(y+z)-[\mu(\beta-\mu-\gamma\Upsilon-\propto y-\propto z)+\beta\Upsilon\gamma\Upsilon]\ [((1-\theta)\beta-\mu-\tau+\propto w)(\beta-\mu-\delta)-\propto\tau w]$

$G(p)=-p^3\{[\mu-(\beta-\mu-\gamma\Upsilon-\propto y-\propto z)]-[(\beta-\mu-\delta)+((1-\theta)\beta-\mu-\tau+\propto w)]\}+p\{[\mu-(\beta-\mu-\gamma\Upsilon-\propto y-\propto z)][((1-\theta)\beta-\mu-\tau+\propto w)(\beta-\mu-\delta)-\propto\tau w]+[\mu(\beta-\mu-\gamma\Upsilon-\propto y-\propto z)+\beta\gamma\Upsilon][(\beta-\mu-\delta)+((1-\theta)\beta-\mu-\tau+\propto w)][\propto(y+z)(\theta\beta-\propto w)(\beta-\mu-\delta)-\mu\propto(y+z)(\theta\beta-\propto w)+\propto^2\tau w(y+z)]\}$

Differentiating with respect to p, we have that

$F^1(p)+4p^3-2p\{[((1-\theta)\beta-\mu-\tau+\propto w)(\beta-\mu-\delta)-\propto\tau w]-[\mu(\beta-\mu-\gamma\Upsilon-\propto y-\propto z)][(\beta-\mu-\delta)+((1-\theta)\beta-\mu-\tau+\propto w)]\ [\mu(\beta-\mu-\gamma\Upsilon-\propto y-\propto z)+\beta\gamma\Upsilon]-\propto(y+z)(\theta\beta-\propto w)\}$

$G^1(p)=-3p^2\{[\mu-(\beta-\mu-\gamma\Upsilon-\propto y-\propto z)]-[(\beta-\mu-\delta)+((1-\theta)\beta-\mu-\tau+\propto w)]\}$

$+[\mu-(\beta-\mu-\gamma\Upsilon-\propto y-\propto z)][((1-\theta)\beta-\mu-\tau+\propto w)(\beta-\mu-\delta)-\propto\tau w]$

$+[\mu(\beta-\mu-\gamma\Upsilon-\propto y-\propto z)+\beta\gamma\Upsilon][(\beta-\mu-\delta)+((1-\theta)\beta-\mu-\tau+\propto w)][\propto(y+z)(\theta\beta-\propto w)(\beta-\mu-\delta)-\mu\propto(y+z)(\theta\beta-\propto w)+\propto^2\tau w(y+z)]$

Setting P = 0, we have

$F(0)=\propto\mu(y+z)(\theta\beta-\propto w)(\beta-\mu-\delta)+\propto^2\mu\tau w(y+z)$

$-[\mu(\beta-\mu-\gamma\Upsilon-\propto(y+z))+\beta\gamma\Upsilon]\ [((1-\theta)\beta-\mu-\tau+\propto w)(\beta-\mu-\delta)-\propto\tau w]$

$G(0)=0$

$F^1(0)=0$

$G^1(0)=[\mu-(\beta-\mu-\gamma\Upsilon-\propto(y+z))]\ [((1-\theta)\beta-\mu-\tau+\propto w)(\beta-\mu-\delta)-\propto\tau w]$

$+[\mu(\beta-\mu-\gamma\Upsilon-\propto(y+z))+\beta\gamma][(\beta-\mu-\delta)+((1-\theta)\beta-\mu-\tau+\propto w)]\ [\propto(y+z)[(\theta\beta-\propto w)(\beta-\mu-\delta)+\propto\tau w]\}$

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The condition for stability accounting to Bellman and Cooke is given by

$F(0) \quad G^1(0) - \quad F^1(0) \quad G(0) \quad > \quad 0$
(27)

Since G(0) = 0, from equation (27)

$F(0)G^1(0) \quad > \quad 0$ ———————————— (28)

The condition for (28) to hold is

Sign $F(0) = \text{sign } G^1(0)$ (29).

Conclusion

The system will be stable at the origin if $\beta < \mu$, that is, the death modulus is higher than the birth modulus, signifying the state of population extinction. The stability of the non zero state which is a state of population sustenance can be attained in meeting the requirement of inequality (28) and equation (29). This will help in policy formulation.

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