A GENERALIZED GAMMA DISTRIBUTION ASSOCIATED WITH $_pF_q$ AND ITS APPLICATION IN RELIABILITY

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ABSTRACT

In this paper a generalized gamma distribution involving $_pF_q$ (generalized hypergeometric function), has been studied from which almost variance. Hazard function have been worked out for generalized gamma distribution having $_pF_q$.

1. INTRODUCTION

In recent years many generalizations of gamma and Weibull distributions are proposed notably by Bradley [1], Srivastava [2], Lee and Gross[4], Bondesson [5]. These generalized distributions are mainly introduced in order to extend the scope of ordering gamma and Weibull distributions and to develop a Model for failure to suit any given particular situation.

Kobayashi [3] has introduced a new type of generalized gamma function as

$$\Gamma_r(m, n) = \int_0^\infty x^{m-1}(x+n)^{-r} e^{-x}\, dx \qquad \dots (1.1)$$

For a positive integer r. Here m and n are parameters of the functions. This function occurs in many problems of diffraction theory [Kobayashi [3]]. However, this generalized gamma function has not yet drawn, the attention of statistician. Agarwal and Kalla [6] has introduced a slightly modified form of the generalized gamma function as

$$\int_0^\infty x^{m-1}(x+n)^{-\lambda} e^{-bx}\, dx = b^{\lambda-m}\, \Gamma_\lambda(m, bn) \qquad \dots (1.2)$$

In this paper we defined a new generalization of gamma distribution involving $_pF_q$ by considering a modified from of the Agarwal and Kalla [6]. A few well known probability distributions are shown to be its particular cases.

FUNCTION OF MATRIX ARGUMENT IN THE COMPLEX CASE :

We consider real valued scalar function of a single matrix argument of the type $\tilde{Z} = \tilde{X} + i\tilde{Y}$ where $\tilde{X}$ and $\tilde{Y}$ are p x p matrices with real elements and $i=\sqrt{-1}$ as well as scaler functions of many matrices $\tilde{Z}_j$, j = 1, 2, …….K where each $\tilde{Z}_j$ is of the type $\tilde{Z}$ above in the real case. We confined our discussion to the situation where the argument matrix was real symmetric positive definite. This was done so that the fractional power of matrices and functions of such matrices could be uniquely defined. Corresponding properties are of we restrict to the class of Hermitian positive definite matrices.

Definition : Hermitian positive definite matrix due to Mathai [11], We will denote the conjugate of $\tilde{Z}$ by $\tilde{\tilde{Z}}$ if $\tilde{Z}$ hermitian, then $\tilde{Z} = \tilde{Z}*$, that is

$$\tilde{Z} = \tilde{Z}* \quad \Rightarrow \quad \tilde{X} + i\tilde{Y} = (\tilde{X} + i\tilde{Y})* = \tilde{X}' + i\tilde{Y}'$$

$$\Rightarrow \quad \tilde{X} = \tilde{X}' \text{ and } \tilde{Y} = \tilde{Y}'$$

Thus $\tilde{X}$ is the symmetric and $\tilde{Y}$ is skew symmetric. Further if $\tilde{Z}$ is hermitian positive definite, then all the eigen values of $\tilde{Z}$ are real and positive. Further, matrix variate gamma in the complex case is

$$\tilde{\Gamma}_p(\alpha) = \pi^{\frac{p(p-1)}{2}\Gamma(\alpha)\Gamma(\alpha-1)\ldots\ldots\Gamma(\alpha-\pi+1)}$$

We will use the notation $\tilde{Z} > 0$ to indicate that $\tilde{Z}$ is hermitian positive definite. Constant matrices will be written without a tilde whether the elements are real or complex unless it has to be emphasized that the matrix involved has complex elements. Then in that case a constant matrix will also be written with a tilde.

ZONAL POLYNOMIAL

Let $\tilde{V}_k$ be the vector space of homogeneous polynomial of degree k, then the Zonal polynomial $\tilde{C}_k(\tilde{X})$ is defined as the component of $(\text{tr}\tilde{X}^k)$ in the subspace $\tilde{V}_k$. $\tilde{C}_k(\tilde{X})$ is also generalization of $\tilde{X}^k$. The exponential function has the following expansion

$$e^{\text{tr}(\tilde{X})} = \sum_{k=0}^{\infty}\frac{1}{k!}\left[\text{tr}(\tilde{X})\right]^k = \sum_{k=0}^{\infty}\sum_{K}\frac{\tilde{C}_K(\tilde{X})}{k!} \qquad \ldots(1.3)$$

The binomial expansion is the following for $I-\tilde{X}>0$ that is $\tilde{X}=\tilde{X}^*>0$ and all eigen values of $\tilde{X}$ are between 0 and 1.

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$$\left|\det\left(I-\tilde{X}\right)\right|^{-\alpha}=\sum_{k=0}^{\infty}\sum_{K}\frac{(\alpha)_K}{k!}\tilde{C}_k(\tilde{X}), \qquad \ldots(1.4)$$

where

$$(\alpha)_K=\prod_{j=1}^{p}\left[\alpha-\frac{j-1}{2}\right]_{k_j},$$

with $K=(k_1,\ldots,k_p), k_1+\ldots+k_p=k$

$$\int_{O(\tilde{P})}\tilde{C}_K(\tilde{H}*\tilde{X}\tilde{H}\tilde{T})d\tilde{H}=\frac{\tilde{C}_K(\tilde{X})\tilde{C}_K(\tilde{T})}{\tilde{C}_K(I)} \qquad \ldots(1.5)$$

where I is the identity matrix, the integral is over the orthogonal group of $p \times p$ matrices and $d\tilde{H}$ is the invariate Harr measure. For detailed study consult Mathai [10].

2. STATISTICAL PROPERTIES OF GENERALIZED GAMMA DISTRIBUTION INVOLVING $_pF_q$

In this section the expression of Mean, Variance, Moment generating function. Hazard function of generalized gamma distribution are discussed.

Theorem 1: If the random variable x follows the generalized gamma distribution involving $_pF_q$ with its respective mean and variance are

$$\text{Mean}=\frac{\Gamma(m+1,bn)_{p+1}F_q\left((\alpha),m+1;(\beta);\left(k/b\right)\right)}{b\Gamma(m,bn)_{p+1}F_q\left((\alpha),m-\lambda;(\beta);k/b\right)}$$

$$\text{Mean}=\frac{m-\lambda}{b}, \quad \text{for small bn} \qquad \ldots(2.1)$$

k=0 or either (α)=0 or (β)=0 and

$$\text{Variance}=\frac{b^{-2}\left(\Gamma_\gamma(m+2,bn)\Gamma_\Gamma(m,bn)_{p+1}F_q(+)-\Gamma_\lambda^2(m+1,bn)_{p+1}F_q(-)\right)}{\Gamma_\lambda(m,bn)_{p+1}F_q(*)}$$

$$=\frac{m-\lambda}{b^2}, \quad \text{for small bn.} \qquad \ldots(2.2)$$

Here $_{p+1}F_q[+]={}_{p+1}F_q\left((\alpha),m+2;(\beta);k/b\right)$

$${}_{p+1}F_q[-] = {}_{p+1}F_q\left((\alpha), m+1; (\beta); k/b\right)$$

$${}_{p+1}F_q[*] = {}_{p+1}F_q\left((\alpha), m-\lambda; (\beta); k/b\right)$$

Proof : By definition

Mean = $E(x) = \int_0^\infty x f(x)dx$

$$= \frac{b^{-1}\Gamma_\lambda(m+1, bn)\,{}_{p+1}F_q\left((\alpha), m+1; (\beta); k/b\right)}{\Gamma_\lambda(m, bn)\,{}_{p+1}F_q\left((\alpha), m-\lambda; (\beta); k/b\right)}$$

Putting k = 0 or either $(\alpha) = 0$ or $(\beta) = 0$

$$= \frac{b^{-1}\Gamma_\lambda(m+1, bn)}{\Gamma_\lambda(m, bn)}$$

$$= \frac{mn\ \ U(m+1, 2+m-\lambda, bn)}{b\ U(m, 1+m-\lambda, bn)}$$

$$= \frac{m-\lambda}{b}, \quad \text{for small bn}$$

and Variance = $E(x^2) - [E(x)]^2$

$$E(x^2) = \int x^2 f(x)dx$$

$$= \frac{\Gamma_\lambda(m+2, bn)\,{}_{p+1}F_q(+)}{b^2\Gamma_\lambda(m, bn)\,{}_{p+1}F_q(*)}$$

Putting k=0, $(\alpha) = 0$ or $(\beta) = 0$

$$= \frac{b^2\Gamma_\lambda(m+2, bn)}{\Gamma_\lambda(m, bn)}$$

$$= \frac{m(m+1)n^2\ \ U(m+2, 3+m-\lambda, bn)}{U(m, 1+m-\lambda, bn)}$$

$$= \frac{(1+m-\lambda)(m-d)}{b^2}, \text{ for small bn}$$

$$\text{Variance} = \frac{m-\lambda}{b^2}$$

Theorem 2 : The r^{th} moment about origin and the moment generating function of random variable x following generalized gamma distribution are

$$\mu_r' = \frac{\Gamma_\lambda(m+r, bn)\,{}_{p+1}F_q\left((\alpha), m+r; (\beta); \left(k/b\right)\right)}{b^r \Gamma_\lambda(m, bn)\,{}_{p+1}F_q\left((\alpha), m-\lambda; (\beta); k/b\right)}$$

and $$\text{M.G.F.} = \frac{\left[1-t/b\right]^{\lambda-m} \Gamma_\lambda\left[(m, bn)(1-t/b)\right]\,{}_{p+1}F_q\left[(\alpha), m; (\beta); k/(b-t)\right]}{\Gamma_\lambda(m, bn)\,{}_{p+1}F_q\left[(\alpha), m-\lambda; (\beta); k/b\right]}$$

Theorem 3 : If x follows the generalized gamma distribution involving ${}_pF_q$ its Hazard or failure rate function is given by

$$H(x) = \frac{e^{-bx} x^{m-1} (x+n)^{-\lambda}\,{}_pF_q[(\alpha); (\beta); kx]}{b^{\lambda-m}\Gamma_\lambda(m, bn, bx)\,{}_{p+1}F_q\left((\alpha), m-\lambda; (\beta); k/b\right)} \quad \ldots(2.3)$$

where $\Gamma_\lambda(m, n, x)$ is generalized in complete gamma function defined as

$$\Gamma_\lambda(m, n, x) = \int_x^\infty \frac{t^{m-1} e^{-t}}{(t+n)^\lambda} dt$$

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