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A GENERALIZED FORMULA FOR CANONICAL POLYNOMIALS FOR m-th ORDER NON-OVERDETERMINED ORDINARY DIFFERENTIAL EQUATIONS (ODEs)

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ABSTRACT

The canonical polynomials play a major role in the Recursive formulation of the Tau method of Lanczos and Ortiz. However,the construction of these polynomials, subsequent to its use, in the tau method is highly demanding and hence polynomial associated with individual DE are often constructed. In this paper, we shall present a derived formula which captures the polynomials for a general class of problems involving non-overdetermined m-th order ODEs (m not fixed). The derivative of this polynomial will also be obtained. For the purpose of validating these results, they are cast as theorems for which mathematical induction principle is employed. We hope to incorporate the results obtained into the tau approximation process for purpose of deriving a general tau approximant of the solution of this class of problems in a future work.

1.0 INTRODUCTION

Differential equations result from physical models of anything that varies - whether in space, in time, in value, in cost, in colour, etc. For example, differential equations exist for models of quantities such as: volume, pressure, temperature, density, magnetization, fracture Strength, dislocation density, chemical potential, etc. These differential equations take the general form

$$y^n = f(x, y, y^{'}, ..., y^{n-1}) \tag{1.1a}$$

where $y^{(i)} \equiv \frac{d^i y}{dx^i}$, $i = 1, 2, ..., n$, is the ith derivative of y with respect to x. The above equation is referred to as nth order differential equation because the highest order derivative appearing in the equation is of order n. A unique soluton $y(x)$ of (1.1a) can be obtained when given supplementary conditions

$$y^{(i)}(0) = \alpha_i, \qquad for \qquad i = 0, 1, ...n - 1 \tag{1.1b}$$

(1.1a) and (1.1b) is referred to as initial value problem (IVP). One of the methods that give an accurate approximate solution to (1.1) is the Tau Method.

The essential of the Tau Method (see Lanczos[11,12] and Ortiz[14]) is to perturb the given differential problem in such a way that its exact solution becomes a polynomial. This is achieved by using a polynomial perturbation term, added to the right hand side of the differential equation. The desired Tau approximation is written in terms of a special polynomial basis, called the canonical polynomial basis, uniquely associated with the given differential operator D (see Ortiz[19]) which defines the given problem. Such basis does not depend on the degree of approximation. The order of the approximation can be increased by just adding one or more canonical polynomials to those already generated and updating the coefficients affecting them.

1.1 LITERATURE REVIEW

To give more flexibility in computation of Tau solution, Lanczos[12] in 1956, introduced a systematic use of the so-called canonical polynomials in the Tau method. A recursive generation of Lanczos canonical polynomials was proposed by Ortiz[14]. The basic approach proposed by Ortiz will be restated briefly in this section as contained in Adeniyi et al[2].

Let $y(x)$ be a known function which satisfies

$$Ly(x) = f(x) \tag{1.2a}$$

where L is an mth order linear differential operator with polynomial coefficients and

$$f(x) = f_0 + f_1 x + f_2 x^2 + ... + f_\sigma x^\sigma$$

is a given polynomial of degree σ with real coefficients f_i, $i = 0(1)\sigma$.
In addition, we assume that $y(x)$, $a \le x \le b$, satisfies the following general linear bondary conditions

$$\sum_{l=1}^{t_j} \sum_{k=0}^{m-1} a_{(lkj)} y^{(k)}(x_{ij}) = \alpha_j, \qquad j = 1(1)m \tag{1.2b}$$

where $y^{(k)}(x_{ij})$, $k = 0(1)m-1$, is the value of $y^{(k)}(x)$ at $x = x_{ij}$, $l = 1(1)t_j$, $j = 1(1)m$; t_j denotes the point of evaluation; a_{lkj}, x_{ij} and α_j are given real numbers. Uniquely associated with operator L in (1.2a) is a sequence $\{Q_r(x)\}$, $r \in N_o S$, of canonical polynomials $Q_r(x)$ such that

$$LQ_r(x) = x^r \tag{1.3}$$

where s is a small finite or empty set of indices with cardinality $s(s \le m+h)$; h is the maximum difference between the exponent r of x and the leading exponent of the generating polynomial Lx^r, for $r \in N_o$. The construction of $Q_r(x)$ using Lx^r is described exhaustively by Ortiz[14]. To apply the constructed polynomials $Q_r(x)$, $r \in N_0$, in the Tau method, we consider the perturbed equation

$$Ly_n(x) = f(x) + H_n(x) \tag{1.4}$$

of (1.2a), where

$$H_n(x) = \sum_{i=0}^{m+s-1} \tau_{m+s-1} T_{n-m+i+1}(x) \tag{1.5}$$

Adopt (1.3) in (1.4) to get

$$Ly_n(x) = \sum_{i=0}^{\sigma} f_i LQ_i(x) + \sum_{i=0}^{m+s-1} \tau_{m+s-1} \sum_{r=0}^{n-m+i+1} C_r^{(n-m+i+1)} LQ_r(x) \tag{1.6}$$

From the linearity of the operator L, we obtain

$$Ly_n(x) = L\left\{ \sum_{i=0}^{\sigma} f_i Q_i(x) + \sum_{i=0}^{m+s-1} \tau_{m+s-1} \sum_{r=0}^{n-m+i+1} C_r^{(n-m+i+1)} Q_r(x) \right\}$$

By assuming that L^{-1} exists and is unique, we have

$$y_n(x) = \sum_{i=0}^{\sigma} f_i Q_i(x) + \sum_{i=0}^{m+s-1} \tau_{m+s-1} \sum_{r=0}^{n-m+i+1} C_r^{(n-m+i+1)} Q_r(x) \tag{1.7}$$

2.0 PROBLEM STATEMENT AND METHODOLOGY

In this paper, we intend to obtain a general formula for the canonical polynomials together with its derivatives for the initial value problem (IVPs) m-th order ordinary differential equation (ODE)

$$Ly(x) := \sum_{r=0}^{m} \left\{ \sum_{k=0}^{N_r} P_{rk} x^k \right\} y^{(r)}(x) = \sum_{r=0}^{F} f_r x^r \tag{2.1a}$$

$$L^* y(x_{rk}) := \sum_{r=0}^{m-1} a_{rk} y^{(r)}(x_{rk}) = \alpha_k, \qquad k = 1(1)m \tag{2.1b}$$

where N_r, F are given non-negative integers and a_{rk}, x_{rk}, α_k, f_r, P_{rk} are given real numbers by seeking an approximant

$$y_n(x) = \sum_{r=0}^{n} a_r x^r, \qquad n < +\infty \tag{2.2}$$

which is the exact solution of the corresponding perturbed problem

$$Ly_n(x) = \sum_{r=0}^{F} f_r x^r + H_n(x) \tag{2.3a}$$

$$L^* y_n(x_{rk}) = \alpha_k, \qquad k = 1(1)m \tag{2.3b}$$

where

$$H_n(x) = \sum_{r=0}^{m+s-1} \tau_{m+s-r} T_{n-m+r+1}(x) \tag{2.4}$$

is the perturbation term. The parameters τ_r, $r = 1(1)m + s$, are to be determined,

$$T_r(x) = Cos\left[rCos^{-1}\left\{ \frac{2x-a-b}{b-a} \right\} \right] \equiv \sum_{k=0}^{r} C_k^{(r)} x^k \tag{2.5}$$

is the Chebyshev polynomial valid in the interval $[a, b]$ (assuming that (2.1) is defined in this interval) and

$$s = max\left\{ N_r - r \,|0 \leq r \leq m \right\} \tag{2.6}$$

2.1 THE GENERALIZED CANONICAL POLYNOMIAL FOR NON-OVERDETERMINED m-th ORDER ODEs

The canonical polynomials for the initial value problems (2.1) will be obtained in this section for cases m=1,2,3, and 4 before the general formula for the case m=m is obtained. Since we are presently considering non-overdetermined problems, s will be zero throughout this section.
The following individual cases are considered and from which the general case will be deduced.

Case $m = 1$:

$$\left(P_{1,0}+P_{1,1}x\right)y'(x)+P_{0,0}y(x)=f(x) \tag{2.7a}$$

$$y(0)=\alpha_0 \tag{2.7b}$$

By Definition (see [1],[6],[7],[8]),

$$x^r = LQ_r(x)$$

From (2.7a), the differential operator L is given by

$$L=\left(P_{1,0}+P_{1,1}x\right)\frac{d}{dx}+P_{1,0}$$

$$Lx^r = rP_{1,0}x^{r-1}+\left(rP_{1,1}+P_{0,0}\right)x^r$$

$$\Rightarrow Lx^r = rP_{1,0}LQ_{r-1}(x)+\left(rP_{1,1}+P_{0,0}\right)LQ_r(x)$$

$$Lx^r = L\left\{rP_{1,0}Q_{r-1}(x)+\left(rP_{1,1}+P_{0,0}\right)Q_r(x)\right\}$$

And due to the existence of L^{-1}, we have

$$x^r = rP_{1,0}Q_{r-1}(x)+\left(rP_{1,1}+P_{0,0}\right)Q_r(x)$$

therefore

$$Q_r(x)=\frac{x^r-rP_{1,0}Q_{r-1}(x)}{rP_{1,1}+P_{0,0}}, \qquad r\geq 0$$

The results for $r = 0, 1, 2, 3$ are

$$Q_0(x)=\frac{1}{P_{0,0}}$$

$$Q_1(x)=\frac{x}{P_{0,0}+P_{1,1}}-\frac{P_{1,0}}{P_{0,0}\left(P_{0,0}+P_{1,1}\right)}$$

$$Q_2(x)=\frac{x^2}{P_{0,0}+2P_{1,1}}-\frac{2P_{1,0}x}{\left(P_{0,0}+P_{1,1}\right)\left(P_{0,0}+2P_{1,1}\right)}+\frac{2P_{1,0}^2}{P_{0,0}\left(P_{0,0}+P_{1,1}\right)\left(P_{0,0}+2P_{1,1}\right)}$$

$$Q_3(x) = \frac{x^3}{P_{0,0}+3P_{1,1}} - \frac{3P_{1,0}x^2}{(P_{0,0}+2P_{1,1})(P_{0,0}+3P_{1,1})} + \frac{6P_{1,0}^2 x}{(P_{0,0}+P_{1,1})(P_{0,0}+2P_{1,1})(P_{0,0}+3P_{1,1})} - \frac{6P_{1,0}^3}{P_{0,0}(P_{0,0}+P_{1,1})(P_{0,0}+2P_{1,1})(P_{0,0}+3P_{1,1})}$$

Case $m = 2$:

$$\left(P_{2,0}+P_{2,1}x+P_{2,2}x^2\right)y^{''}(x) + (P_{1,0}+P_{1,1}x)\,y^{'}(x) + P_{0,0}y(x) = f(x) \tag{2.8a}$$

$$y^{'}(0) = \alpha_1, \qquad y(0) = \alpha_2 \tag{2.8b}$$

From (2.8a),

$$L = \left(P_{2,0}+P_{2,1}x+P_{2,2}x^2\right)\frac{d^2}{dx^2} + (P_{1,0}+P_{1,1}x)\frac{d}{dx} + P_{0,0}$$

Passing through the same process as in the case m=0, we have

$$Q_r(x) = \frac{x^r - [r(r-1)P_{2,1}+rP_{1,0}]\,Q_{r-1}(x) - r(r-1)P_{2,0}Q_{r-2}(x)}{r(r-1)P_{2,2}+rP_{1,1}+P_{0,0}}, \qquad r \geq 0$$

From which we obtain $Q_r(x)$ for $r = 0, 1, 2, 3$ as

$$Q_0(x) = \frac{1}{P_{0,0}}$$

$$Q_1(x) = \frac{x}{P_{0,0}+P_{1,1}} - \frac{P_{1,0}}{P_{0,0}(P_{0,0}+P_{1,1})}$$

$$Q_2(x) = \frac{x^2}{P_{0,0}+2P_{1,1}} - \frac{2P_{1,0}x}{(P_{0,0}+P_{1,1})(P_{0,0}+2P_{1,1})} + \frac{2P_{1,0}^2}{P_{0,0}(P_{0,0}+P_{1,1})(P_{0,0}+2P_{1,1})}$$

$$Q_3(x) = \frac{x^3}{P_{0,0}+3P_{1,1}} - \frac{3P_{1,0}x^2}{(P_{0,0}+2P_{1,1})(P_{0,0}+3P_{1,1})} + \frac{6P_{1,0}^2 x}{(P_{0,0}+P_{1,1})(P_{0,0}+2P_{1,1})(P_{0,0}+3P_{1,1})} - \frac{6P_{1,0}^3}{P_{0,0}(P_{0,0}+P_{1,1})(P_{0,0}+2P_{1,1})(P_{0,0}+3P_{1,1})}$$

Case $m = 3$:

$$\left(P_{3,0} + P_{3,1}x + P_{3,2}x^2 + P_{3,3}x^3\right) y^{'''}(x) + \left(P_{2,0} + P_{2,1}x + P_{2,2}x^2\right) y^{''}(x) + \left(P_{1,0} + P_{1,1}x\right) y^{'}(x) + P_{0,0}y(x) = f(x) \quad (2.9a)$$

$$y(0) = \alpha_0, \qquad y^{'}(0) = \alpha_1, \qquad y^{''}(0) = \alpha_2 \qquad (2.9b)$$

From (2.9a), we obtain

$$Q_r(x) = \frac{x^r - r(r-1)(r-2)P_{3,0}Q_{r-3}(x)}{r(r-1)(r-2)P_{3,3} + r(r-1)P_{2,2} + rP_{1,1} + P_{0,0}} - \frac{\left[r(r-1)(r-2)P_{3,1} + r(r-1)P_{2,0}\right]Q_{r-2}(x)}{r(r-1)(r-2)P_{3,3} + r(r-1)P_{2,2} + rP_{1,1} + P_{0,0}} - \frac{\left[r(r-1)(r-2)P_{3,2} + r(r-1)P_{2,1} + rP_{1,0}\right]Q_{r-1}(x)}{r(r-1)(r-2)P_{3,3} + r(r-1)P_{2,2} + rP_{1,1} + P_{0,0}}, \qquad r \geq 0$$

From which we obtain $Q_r(x)$ for $r = 0, 1, 2, 3$ as

$$Q_0(x) = \frac{1}{P_{0,0}}$$

$$Q_1(x) = \frac{x}{P_{0,0} + P_{1,1}} - \frac{P_{1,0}}{P_{0,0}\left(P_{0,0} + P_{1,1}\right)}$$

$$Q_2(x) = \frac{x^2}{P_{0,0} + 2P_{1,1} + 2P_{2,2}} - \frac{\left(2P_{2,1} + 2P_{1,0}\right)x}{\left(P_{0,0} + P_{1,1}\right)\left(P_{0,0} + 2P_{1,1} + 2P_{2,2}\right)} + \frac{P_{1,0}\left(2P_{2,1} + 2P_{1,0}\right)}{P_{0,0}\left(P_{0,0} + P_{1,1}\right)\left(P_{0,0} + 2P_{1,1} + 2P_{2,2}\right)} - \frac{2P_{2,0}}{P_{0,0}\left(P_{0,0} + 2P_{1,1} + 2P_{2,2}\right)}$$

Case $m = 4$:

$$\left(P_{4,0} + P_{4,1}x + P_{4,2}x^2 + P_{4,3}x^3 + P_{4,4}x^4\right) y^{(\mathrm{iv})}(x) + \left(P_{3,0} + P_{3,1}x + P_{3,2}x^2 + P_{3,3}x^3\right) y'''(x) + \left(P_{2,0} + P_{2,1}x + P_{2,2}x^2\right) y''(x) + \left(P_{1,0} + P_{1,1}x\right) y'(x) + P_{0,0}y(x) = f(x) \qquad (2.10a)$$

$$y(0) = \alpha_0, \qquad y'(0) = \alpha_1, \qquad y''(0) = \alpha_2, \qquad y'''(0) = \alpha_3 \qquad (2.10b)$$

This yields the canonical polynomial

$$Q_r(x) =$$
$$\frac{x^r}{r(r-1)(r-2)(r-3)P_{4,4}+r(r-1)(r-2)P_{3,3}+r(r-1)P_{2,2}+rP_{1,1}+P_{0,0}}$$
$$-\frac{[r(r-1)(r-2)(r-3)P_{4,3}+r(r-1)(r-2)P_{3,2}+r(r-1)P_{2,1}+rP_{1,0}]\,Q_{r-1}(x)}{r(r-1)(r-2)(r-3)P_{4,4}+r(r-1)(r-2)P_{3,3}+r(r-1)P_{2,2}+rP_{1,1}+P_{0,0}}$$
$$-\frac{[r(r-1)(r-2)(r-3)P_{4,2}+r(r-1)(r-2)P_{3,1}+r(r-1)P_{2,0}]\,Q_{r-2}(x)}{r(r-1)(r-2)(r-3)P_{4,4}+r(r-1)(r-2)P_{3,3}+r(r-1)P_{2,2}+rP_{1,1}+P_{0,0}}$$
$$-\frac{[r(r-1)(r-2)(r-3)P_{4,1}+r(r-1)(r-2)P_{3,0}]\,Q_{r-3}(x)}{r(r-1)(r-2)(r-3)P_{4,4}+r(r-1)(r-2)P_{3,3}+r(r-1)P_{2,2}+\mathrm{rP}_{1,1}+P_{0,0}}$$
$$-\frac{r(r-1)(r-2)(r-3)P_{4,0}Q_{r-4}(x)}{r(r-1)(r-2)(r-3)P_{4,4}+r(r-1)(r-2)P_{3,3}+r(r-1)P_{2,2}+\mathrm{rP}_{1,1}+P_{0,0}}$$

For $r = 0, 1, 2,$

$$Q_0(x) = \frac{1}{P_{0,0}}$$

$$Q_1(x) = \frac{x}{P_{0,0}+P_{1,1}} - \frac{P_{1,0}}{P_{0,0}\,(P_{0,0}+P_{1,1})}$$

$$Q_2(x) = \frac{x^2}{P_{0,0}+2P_{1,1}+2P_{2,2}} - \frac{2\,(P_{1,0}+P_{2,1})\,x}{(P_{0,0}+P_{1,1})\,(P_{0,0}+2P_{1,1}+2P_{2,2})} + \frac{2\left(P_{1,0}^2-(P_{0,0}+P_{1,1})\,P_{2,0}+P_{1,0}P_{2,1}\right)}{P_{0,0}\,(P_{0,0}+P_{1,1})\,(P_{0,0}+2\,(P_{1,1}+P_{2,2}))}$$

Hence , we obtained the canonical polynomials formula of the m-th order ODEs as

$$Q_r(x) = \frac{x^r - \sum_{k=1}^{m}\sum_{j=k}^{m}\left(j!\binom{r}{j}P_{j,j-k}\right)Q_{r-k}(x)}{\sum_{k=0}^{m}k!\binom{r}{k}P_{k,k}}, \qquad r \geq 0 \qquad (2.11)$$

<u>Theorem</u>

If the linear differential operator associated with the m-th order linear ODE:

$$Ly(x) = P_0(x)y^{(0)}(x) + ... + P_m(x)y^{(m)}(x) = f(x)$$

where $P_k(x)$ is a k-th degree polynomials, $y^{(k)}(x)$ is the kth derivative of $y(x)$ and $f(x)$ is a polynomial function, is given by

$$L = P_m(x)\frac{d^m}{dx^m} + P_{m-1}(x)\frac{d^{m-1}}{dx^{m-1}} + ... + P_0(x)$$

then

$$Q_r(x) = \frac{x^r - \sum_{k=1}^{m}\sum_{j=k}^{m}\left(j!\binom{r}{j}P_{j,j-k}\right)Q_{r-k}(x)}{\sum_{k=0}^{m}k!\binom{r}{k}P_{k,k}}, \qquad r \geq 0 \qquad (2.12)$$

Proof

We shall employ the principles of mathematical induction to establish the validity of our $Q_r(x)$. If we fix $r = 1$, then we can apply the principles of mathematical induction on m:

$$Q_1(x) = \frac{x - \sum_{k=1}^{m} \sum_{j=k}^{m} \left(j! \binom{1}{j} P_{j,j-k} \right) Q_{1-k}(x)}{\sum_{k=0}^{m} k! \binom{1}{k} P_{k,k}} \tag{2.13}$$

We try for $m = 1$,

$$Q_1(x) = \frac{x - 1!\binom{1}{1} P_{1,0} Q_0(x)}{0!\binom{1}{0} P_{0,0} + 1!\binom{1}{1} P_{1,1}}$$

$$\Rightarrow Q_1(x) = \frac{x - P_{1,0} Q_0(x)}{P_{0,0} + P_{1,1}} \tag{2.14}$$

and this is the same as our $Q_1(x)$ obtained in (2.7) above, confirming that it is true for $m = 1$.Now assume that (2.11) is true for $m = n$, thus (2.11) becomes

$$Q_1(x) = \frac{x - \sum_{k=1}^{n} \sum_{j=k}^{n} \left(j! \binom{1}{j} P_{j,j-k} \right) Q_{1-k}(x)}{\sum_{k=0}^{n} k! \binom{1}{k} P_{k,k}} \tag{2.15}$$

The next thing is to show that (2.11) holds for $m = n + 1$. From our construction of $Q_1(x)$ in (2.13) and (2.15) for $m = n$ to $m = n + 1$, we have

$$Q_1(x) = \frac{x - \left\{ \sum_{k=1}^{n} \sum_{j=k}^{n} \left(P_{j,j-k} \right) j! \binom{1}{j} \right\} Q_{1-k}}{\sum_{k=0}^{n} k! \binom{1}{k} P_{k,k} + (n+1)! \binom{1}{n+1} P_{n+1,n+1}} + \frac{\left\{ \left(P_{n+1,n-k+1} \right) (n+1)! \binom{1}{n+1} \right\} Q_{1-k}}{\sum_{k=0}^{n} k! \binom{1}{k} P_{k,k} + (n+1)! \binom{1}{n+1} P_{n+1,n+1}} \tag{2.16}$$

$$Q_1(x) = \frac{x - \left\{ \sum_{k=1}^{n} \sum_{j=k}^{n} \left(P_{j,j-k} \right) j! \binom{1}{j} + \left(P_{n+1,n-k+1} \right) (n+1)! \binom{1}{n+1} \right\} Q_{1-k}}{\sum_{k=0}^{n+1} (P_{k,k}) k! \binom{1}{k}} \tag{2.17}$$

$$Q_1(x) = \frac{x - \left\{ \sum_{k=1}^{n} \sum_{j=k}^{n} \left(P_{j,j-k} \right) j! \binom{1}{j} + \left(P_{n+1,n-k+1} \right) (n+1)! \binom{1}{n+1} \right\} Q_{1-k}}{\sum_{k=0}^{n+1} (P_{k,k}) k! \binom{1}{k}} \tag{2.18}$$

Thus, it is true for $m = n + 1$ also.

If we choose to fix $r = l$, so that we can again apply the principle of mathematical induction on m,

$$Q_l(x) = \frac{x^l - \sum_{k=1}^{m} \sum_{j=k}^{m} \left(j! \binom{1}{j} P_{j,j-k} \right) Q_{l-k}(x)}{\sum_{k=0}^{m} k! \binom{1}{k} P_{k,k}}, \qquad r \geq 0 \tag{2.19}$$

We try for $m = 1$,

$$Q_l(x) = \frac{x^l - 1!\binom{1}{1}P_{1,0}Q_{l-1}(x)}{0!\binom{1}{0}P_{0,0} + 1!\binom{1}{1}P_{1,1}} \tag{2.20}$$

$$Q_l(x) = \frac{x^l - lP_{1,0}Q_{l-1}(x)}{P_{0,0} + lP_{1,1}} \tag{2.21}$$

and this is in conformity with our earlier results, thus, it is true for $m = 1$.

Now, assume that (2.19) is true for $m = n$, thus (2.19) becomes

$$Q_l(x) = \frac{x^l - \sum_{k=1}^{n}\sum_{j=k}^{n}\left(j!\binom{1}{j}P_{j,j-k}\right)Q_{l-k}(x)}{\sum_{k=0}^{n} k!\binom{1}{k}P_{k,k}} \tag{2.22}$$

The next thing is to prove that (2.19) holds for $m = n + 1$.

From our construction of $Q_l(x)$ for $m = n$ up to $m = n + 1$, we have

$$\begin{aligned} Q_l(x) = {} & \frac{x^l - \left\{\sum_{k=1}^{n}\sum_{j=k}^{n}\left(P_{j,j-k}\right)j!\binom{1}{j}\right\}Q_{l-k}(x)}{\sum_{k=0}^{n} k!\binom{1}{k}P_{k,k} + (n+1)!\binom{1}{n+1}P_{n+1,n+1}} \\ & + \frac{\left\{\left(P_{n+1,n-k+1}\right)(n+1)!\binom{1}{n+1}\right\}Q_{1-k}(x)}{\sum_{k=0}^{n} k!\binom{1}{k}P_{k,k} + (n+1)!\binom{1}{n+1}P_{n+1,n+1}} \end{aligned} \tag{2.23}$$

$$Q_l(x) = \frac{x^l - \left\{\sum_{k=1}^{n}\sum_{j=k}^{n}\left(P_{j,j-k}\right)j!\binom{1}{j} + \left(P_{n+1,n-k+1}\right)(n+1)!\binom{1}{n+1}\right\}Q_{l-k}}{\sum_{k=0}^{n+1}(P_{k,k})k!\binom{1}{k}} \tag{2.24}$$

$$Q_l(x) = \frac{x^l - \sum_{k=1}^{n+1}\sum_{j=k}^{n+1}\left(j!\binom{1}{j}P_{j,j-k}\right)Q_{l-k}(x)}{\sum_{k=0}^{n+1} k!\binom{1}{k}P_{k,k}} \tag{2.25}$$

Thus it is true for $m = n + 1$.

Finally, if we choose to fix $r = l+1$, and we apply the principle of mathematical induction on m. With $r = l + 1$, (2.11) is now

$$Q_{l+1}(x) = \frac{x^{l+1} - \sum_{k=1}^{m}\sum_{j=k}^{m}\left(j!\binom{l+1}{j}P_{j,j-k}\right)Q_{l-k+1}(x)}{\sum_{k=0}^{m} k!\binom{l+1}{k}P_{k,k}} \tag{2.26}$$

We try for $m = 1$

,

$$Q_{l+1}(x) = \frac{x^{l+1} - 1!\binom{l+1}{1}P_{1,0}Q_l(x)}{0!\binom{l+1}{0}P_{0,0} + 1!\binom{l+1}{1}P_{1,1}} \tag{2.27}$$

$$Q_{l+1}(x) = \frac{x^{l+1} - (l+1)Q_l(x)}{P_{0,0} + (l+1)P_{1,1}} \tag{2.28}$$

which shows it is true for $m = 1$.

We assume it is true for $m = n$ so that (2.26), with $r = l + 1$, becomes

$$Q_{l+1}(x) = \frac{x^{l+1} - \sum_{k=1}^{n} \sum_{j=k}^{n} \left(j!\binom{l+1}{j} P_{j,j-k} \right) Q_{l-k+1}(x)}{\sum_{k=0}^{n} k!\binom{l+1}{k} P_{k,k}} \tag{2.29}$$

The next thing is to prove that (2.26) holds for $m = n + 1$,

$$\begin{aligned} Q_{l+1}(x) = {} & \frac{x^{l+1} - \left\{ \sum_{k=1}^{n} \sum_{j=k}^{n} \left(P_{j,j-k} \right) j!\binom{l+1}{j} \right\} Q_{l-k+1}(x)}{\sum_{k=0}^{n} k!\binom{l+1}{k} P_{k,k} + (n+1)!\binom{l+1}{n+1} P_{n+1,n+1}} \\ & + \frac{\left\{ \left(P_{n+1,n-k+1} \right) (n+1)!\binom{l+1}{n+1} \right\} Q_{1-k+1}(x)}{\sum_{k=0}^{n} k!\binom{l+1}{k} P_{k,k} + (n+1)!\binom{l+1}{n+1} P_{n+1,n+1}} \end{aligned} \tag{2.30}$$

$$\begin{aligned} Q_{(l+1)}(x) = {} & \frac{x^{l+1} - \sum_{k=1}^{n} \sum_{j=k}^{n} \left(P_{j,j-k} \right) j!\binom{l+1}{j}}{\sum_{k=0}^{n+1} (P_{k,k}) k!\binom{l+1}{k}} \\ & + \frac{\left(P_{n+1,n-k+1} \right) (n+1)!\binom{l+1}{n+1} Q_{l-k+1}}{\sum_{k=0}^{n+1} (P_{k,k}) k!\binom{l+1}{k}} \end{aligned} \tag{2.31}$$

$$Q_{l+1}(x) = \frac{x^{l+1} - \sum_{k=1}^{n+1} \sum_{j=k}^{n+1} \left(j!\binom{l+1}{j} P_{j,j-k} \right) Q_{l-k+1}(x)}{\sum_{k=0}^{n+1} k!\binom{1}{k} P_{k,k}}, \tag{2.32}$$

Thus, the formula for $Q_r(x)$ given by (2.11) holds for all r and for all m. Hence, its validation.

3.0 THE DERIVATIVES OF m-th ORDER DIFFERENTIAL EQUATION

The method of section 2 is also applied here to obtain a general result for the derivative of the canonical polynomial. So doing, we considered the individual cases below:

Case $m = 1$:
First Derivative:

$$Q'_r(x) = \frac{rx^{r-1} - rP_{1,0}Q'_{r-1}(x)}{P_{0,0} + rP_{1,1}} \tag{3.1}$$

Second Derivative:

$$Q''_r(x) = \frac{r(r-1)x^{r-2} - rP_{1,0}Q''_{r-1}(x)}{P_{0,0} + rP_{1,1}} \tag{3.2}$$

Third Derivative:

$$Q'''_r(x) = \frac{r(r-1)(r-2)x^{r-3} - rP_{1,0}Q'''_{r-1}(x)}{P_{0,0} + rP_{1,1}} \tag{3.3}$$

If we continue with this process, we shall obtain the nth derivative for case $m = 1$ as

$$Q_r^{(n)}(x) = \frac{n!\binom{r}{n}x^{r-n} - rP_{1,0}Q_{r-1}^{(n)}(x)}{P_{0,0} + rP_{1,1}} \tag{3.4}$$

Following a similar procedure, the nth derivatives for cases $m = 2$ and $m = 3$ are

$$Q_r^{(n)}(x) = \frac{n!\binom{r}{n}x^{r-n} - r(r-1)P_{2,1}Q_{r-1}^{(n)}(x) - r(r-1)P_{2,0}Q_{r-2}^{(n)}(x)}{P_{0,0} + rP_{1,1} + r(r-1)P_{2,2}} \tag{3.5}$$

and

$$Q_r^{(n)}(x) = \frac{n!\binom{r}{n}x^{r-n} - \left[r(r-1)(r-2)P_{3,2} + r(r-1)P_{2,1} + rP_{1,0}\right]Q_{r-1}^{(n)}(x)}{P_{0,0} + rP_{1,1} + r(r-1)P_{2,2} + r(r-1)(r-2)P_{3,3}}$$
$$- \frac{\left[r(r-1)(r-2)P_{3,1} + r(r-1)P_{2,0}\right]Q_{r-2}^{(n)}(x) + r(r-1)(r-2)P_{3,0}Q_{r-3}^{(n)}(x)}{P_{0,0} + rP_{1,1} + r(r-1)P_{2,2} + r(r-1)(r-2)P_{3,3}} \tag{3.6}$$

respectively.
We deduce from that for general m-th order equation that the n-th derivative of $Q_r(x)$ is

$$Q_r^{(n)}(x) = \frac{n!\binom{r}{n}x^{r-n} - \sum_{k=1}^{m}\left(\sum_{j=k}^{m} j!\binom{r}{j}P_{j,j-k}\right)Q_{r-k}^{(n)}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k}} \tag{3.7}$$

Theorem

If the canonical polynomials associated with the m-th order linear DE:

$$Ly(x) \equiv P_0(x)y^{(0)}(x) + ... + P_m(x)y^{(m)}(x) = f(x)$$

where $P_k(x)$ is a k-th degree polynomial, $y^{(k)}(x)$ is the kth derivative of $y(x)$ and $f(x)$ is a polynomial function, is given by

$$Q_r(x) = \frac{x^r - \sum_{k=1}^{m}\left(\sum_{j=k}^{m} j!\binom{r}{j}P_{j,j-k}\right)Q_{r-k}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k}}$$

then

$$Q_r^{(n)}(x) = \frac{n!\binom{r}{n}x^{r-n} - \sum_{k=1}^{m}\left(\sum_{j=k}^{m} j!\binom{r}{j}P_{j,j-k}\right)Q_{r-k}^{(n)}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k}} \tag{3.8}$$

Proof

We shall establish this by the principle of mathematical induction. We start by fixing $r = 1$ and subject m to varied values:

Let us try for $m = 1$,

$$Q_1^{(n)}(x) = \frac{n!\binom{1}{n}x^{1-n} - 1!\{1}{1}P_{1,0}Q_0^{(n)}(x)0!\binom{1}{0}P_{0,0} + 1!\binom{1}{1}P_{1,1} \tag{3.9}$$

$$Q_1^{(n)}(x) = \frac{n!\binom{1}{n}x^{1-n} - P_{1,0}Q_0^{(n)}(x)}{P_{0,0} + P_{1,1}} \tag{3.10}$$

which shows that it is true for $m = 1$, since this (i.e 3.10) is in conformity with (3.8) when $r = 1$.

We assume that it is true for $m = q$,

$$Q_r^{(n)}(x) = \frac{n!\binom{r}{n}x^{r-n} - \sum_{k=1}^{q}\left(\sum_{j=k}^{q} j!\binom{r}{j}P_{j,j-k}\right)Q_{r-k}^{(n)}(x)}{\sum_{k=0}^{q} k!\binom{r}{k}P_{k,k}} \tag{3.11}$$

Now we shall prove that (3.8) holds for $m = q + 1$.

From our construction of $Q_1^{(n)}(x)$ in (3.8) (with $r = 1$) for $m = q$ up to $m = q + 1$, we have

$$Q_l^{(n)}(x) = \frac{n!\binom{1}{n}x^{1-n} - \left(\sum_{k=1}^{q}\sum_{j=k}^{q}\left(P_{j,j-k}\right)j!\binom{1}{j}\right)Q_{1-k}^{(n)}(x)}{\sum_{k=0}^{q}k!\binom{1}{k}P_{k,k} + (q+1)!\binom{1}{q+1}P_{q+1,q+1}} + \frac{\left\{\left(P_{q+1,q-k+1}\right)(q+1)!\binom{1}{q+1}\right\}Q_{1-k}^{(n)}(x)}{\sum_{k=0}^{q}k!\binom{1}{k}P_{k,k} + (q+1)!\binom{1}{q+1}P_{q+1,q+1}} \tag{3.12}$$

$$Q_l^{(n)}(x) = \frac{n!\binom{1}{n}x^{1-n} - \left\{\sum_{k=1}^{q}\sum_{j=k}^{q}\left(P_{j,j-k}\right)j!\binom{1}{j}\right\}Q_{1-k}^{(n)}(x)}{\sum_{k=0}^{q+1}k!\binom{1}{k}P_{k,k}} + \frac{\left\{\left(P_{q+1,q-k+1}\right)(q+1)!\binom{1}{q+1}\right\}Q_{1-k}^{(n)}(x)}{\sum_{k=0}^{q+1}k!\binom{1}{k}P_{k,k}} \tag{3.13}$$

$$Q_l^{(n)}(x) = \frac{n!\binom{1}{n}x^{1-n} - \sum_{k=1}^{q+1}\sum_{j=k}^{q+1}\left(P_{j,j-k}\right)j!\binom{1}{j}Q_{1-k}^{(n)}(x)}{\sum_{k=0}^{q+1}k!\binom{1}{k}P_{k,k}} \tag{3.14}$$

Thus, it is true for $m = q + 1$ also.

If we decide to fix $r = l$, so that we can apply the principle of mathematical induction on m again, we have

$$Q_l^{(n)}(x) = \frac{n!\binom{1}{n}x^{l-n} - \sum_{k=1}^{m}\sum_{j=k}^{m}\left(P_{j,j-k}\right)j!\binom{1}{j}Q_{l-k}^{(n)}(x)}{\sum_{k=0}^{m}k!\binom{1}{k}P_{k,k}} \tag{3.15}$$

We try for $m = 1$,

$$Q_l^{(n)}(x) = \frac{n!\binom{1}{n}x^{l-n} - 1!\binom{1}{1}P_{1,0}Q_{l-1}^{(n)}(x)}{0!\binom{1}{0}P_{0,0} + 1!\binom{1}{1}P_{1,1}} \tag{3.16}$$

$$Q_l^{(n)}(x) = \frac{n!\binom{1}{n}x^{l-n} - lP_{1,0}Q_{l-1}^{(n)}(x)}{P_{0,0} + lP_{1,1}} \tag{3.17}$$

Since this tallies with (3.8) with r replaced by l, thus it is true for $m = 1$.

Let us assume that (3.8) holds for $m = q$. Now to prove that (3.8) holds for $m = q + 1$.

From our construction of $Q_l^{(n)}(x)$ in (3.8) for $m = q$ up to $m = q+1$, we have

$$Q_l^{(n)}(x) = \frac{n!\binom{1}{n}x^{l-n} - \left\{\sum_{k=1}^{q}\sum_{j=k}^{q}\left(P_{j,j-k}\right)j!\binom{1}{j}\right\}Q_{l-k}^{(n)}(x)}{\sum_{k=0}^{q}k!\binom{1}{k}P_{k,k} + (q+1)!\binom{1}{q+1}P_{q+1,q+1}} + \frac{\left\{\left(P_{q+1,q-k+1}\right)(q+1)!\binom{1}{q+1}\right\}Q_{l-k}^{(n)}(x)}{\sum_{k=0}^{q}k!\binom{1}{k}P_{k,k} + (q+1)!\binom{1}{q+1}P_{q+1,q+1}} \tag{3.18}$$

$$Q_l^{(n)}(x) = \frac{n!\binom{1}{n}x^{l-n}}{\sum_{k=0}^{q+1}\left(P_{k,k}\right)k!\binom{1}{k}} - \frac{\sum_{k=1}^{q}\sum_{j=k}^{q}\left\{j!\binom{1}{j}\left(P_{j,j-k}\right) + (q+1)!\binom{1}{q+1}\left(P_{q+1,q+1}\right)\right\}Q_{l-k}^{(n)}(x)}{\sum_{k=0}^{q+1}\left(P_{k,k}\right)k!\binom{1}{k}} \tag{3.19}$$

$$Q_l^{(n)}(x) = \frac{n!\binom{1}{n}x^{l-n} - \sum_{k=1}^{q+1}\left(\sum_{j=k}^{q+1}j!\binom{1}{j}P_{j,j-k}\right)Q_{l-k}^{(n)}(x)}{\sum_{k=0}^{q}k!\binom{1}{k}P_{k,k}} \tag{3.20}$$

Thus it is true for $m = q+1$

Finally, if we choose to fix $r = l+1$, and we apply the principle of mathematical induction on m. With our $r = l+1$, (3.15) becomes

$$Q_{l+1}^{(n)}(x) = \frac{n!\binom{l+1}{n}x^{l-n+1} - \sum_{k=1}^{m}\sum_{j=k}^{m}\left(P_{j,j-k}\right)j!\binom{l+1}{j}Q_{l-k+1}^{(n)}(x)}{\sum_{k=0}^{m}k!\binom{l+1}{k}P_{k,k}} \tag{3.21}$$

We try for $m = 1$,

$$Q_{(l+1)}^{(n)}(x) = \frac{n!\binom{l+1}{n}x^{l-n+1} - 1!\binom{l+1}{1}P_{1,0}Q_l^{(n)}(x)}{0!\binom{l+1}{0}P_{0,0} + 1!\binom{l+1}{1}P_{1,1}} \tag{3.22}$$

$$Q_{(l+1)}^{(n)}(x) = \frac{n!\binom{l+1}{n}x^{l-n+1} - (l+1)P_{1,0}Q_l^{(n)}(x)}{P_{0,0} + (l+1)P_{1,1}} \tag{3.23}$$

and this is the same as (3.8) with r replaced $l+1$, confirming that it is true for $m = 1$.

We assume that it is true for $m = q$, thus (3.21) can now be written as:

$$Q_{l+1}^{(n)}(x) = \frac{n!\binom{l+1}{n}x^{l-n+1} - \sum_{k=1}^{q}\left(\sum_{j=k}^{q}j!\binom{l+1}{j}P_{j,j-k}\right)Q_{l-k+1}^{(n)}(x)}{\sum_{k=0}^{q}k!\binom{l+1}{k}P_{k,k}} \tag{3.24}$$

The next stage of our work is to prove that it is true for $m = q + 1$.

From our construction of $Q_{l+1}^{(n)}(x)$ in (3.21) for $m = q$ up to $m = q + 1$, we have

$$Q_{l+1}^{(n)}(x) = \frac{n!\binom{l+1}{n}x^{l-n+1} - \left\{\sum_{k=1}^{q}\sum_{j=k}^{q}\left(P_{j,j-k}\right)j!\binom{l+1}{j}\right\}Q_{l-k+1}^{(n)}(x)}{\sum_{k=0}^{q}k!\binom{l+1}{k}P_{k,k} + (q+1)!\binom{l+1}{q+1}P_{q+1,q+1}} + \frac{\left\{\left(P_{q+1,q-k+1}\right)(q+1)!\binom{l+1}{q+1}\right\}Q_{l-k+1}^{(n)}(x)}{\sum_{k=0}^{q}k!\binom{l+1}{k}P_{k,k} + (q+1)!\binom{l+1}{q+1}P_{q+1,q+1}} \tag{3.25}$$

$$Q_{l+1}^{(n)}(x) = \frac{n!\binom{l+1}{n}x^{l-n+1}}{\sum_{k=0}^{q+1}\left(P_{k,k}\right)k!\binom{l+1}{k}} - \frac{\sum_{k=1}^{q}\sum_{j=k}^{q}\left\{j!\binom{l+1}{j}\left(P_{j,j-k}\right) + (q+1)!\binom{l+1}{q+1}\left(P_{q+1,q-k+1}\right)\right\}Q_{l-k+1}^{(n)}(x)}{\sum_{k=0}^{q+1}\left(P_{k,k}\right)k!\binom{l+1}{k}} \tag{3.26}$$

$$Q_{l+1}^{(n)}(x) = \frac{n!\binom{l+1}{n}x^{l-n+1} - \sum_{k=1}^{q+1}\left(\sum_{j=k}^{q+1}j!\binom{l+1}{j}P_{j,j-k}\right)Q_{l-k+1}^{(n)}(x)}{\sum_{k=0}^{q+1}k!\binom{l+1}{k}P_{k,k}} \tag{3.27}$$

From the above, we can conveniently say that (3.8) holds for all calues of m and r.

We now want to fix our n at $n = 1$ and apply the principle of mathematical induction on m.

With $n = 1$, (3.8) becomes

$$Q_r'(x) = \frac{1!\binom{r}{1}x^{r-1} - \sum_{k=1}^{m}\left(\sum_{j=k}^{m}j!\binom{r}{j}P_{j,j-k}\right)Q_{r-k}'(x)}{\sum_{k=0}^{m}k!\binom{r}{k}P_{k,k}} \tag{3.28}$$

We try for $m = 1$,

$$Q_r'(x) = \frac{1!\binom{r}{1}x^{r-1} - 1!\binom{r}{1}P_{1,0}Q_{r-1}'(x)}{0!\binom{r}{0}P_{0,0} + 1!\binom{r}{1}P_{1,1}} \tag{3.29}$$

$$Q_r'(x) = \frac{rx^{r-1} - rP_{1,0}Q_{r-1}'(x)}{P_{0,0} + rP_{1,1}} \tag{3.30}$$

which is the same as (3.4), confirming the correctness of (3.36), that is, it is true for $m = 1$.

We assume that it is true for $m = q$, that is (3.28) becomes

$$Q_r^{'}(x) = \frac{rx^{r-1} - \sum_{k=1}^{q} \sum_{j=k}^{q} \left(j!\binom{r}{j} P_{j,j-k} \right) Q_{r-k}^{'}(x)}{\sum_{k=0}^{q} k!\binom{r}{k} P_{k,k}} \tag{3.31}$$

Next is to prove that it is true for $m = q + 1$,

$$Q_r^{'}(x) = \frac{rx^{r-1} - \sum_{k=1}^{q} \sum_{j=k}^{q} \left\{ j!\binom{r}{j} P_{j,j-k} \right\} Q_{r-k}^{'}(x)}{\sum_{k=0}^{q} k!\binom{r}{k} P_{k,k} + (q+1)!\binom{r}{q+1} P_{q+1,q+1}} + \frac{\left\{ \left(P_{q+1,q-k+1}\right)(q+1)!\binom{r}{q+1} \right\} Q_{r-k}^{'}(x)}{\sum_{k=0}^{q} k!\binom{r}{k} P_{k,k} + (q+1)!\binom{r}{q+1} P_{q+1,q+1}} \tag{3.32}$$

$$Q_r^{'}(x) = \frac{rx^{r-1}}{\sum_{k=0}^{q} k!\binom{r}{k} P_{k,k} + (q+1)!\binom{r}{q+1} P_{q+1,q+1}} - \frac{\sum_{k=1}^{q} \sum_{j=k}^{q} \left\{ j!\binom{r}{j} P_{j,j-k} + \left(P_{q+1,q-k+1}\right)(q+1)!\binom{r}{q+1} \right\} Q_{r-k}^{'}(x)}{\sum_{k=0}^{q} k!\binom{r}{k} P_{k,k} + (q+1)!\binom{r}{q+1} P_{q+1,q+1}} \tag{3.33}$$

$$Q_r^{'}(x) = \frac{rx^{r-1} - \sum_{k=1}^{q+1} \left(\sum_{j=k}^{q+1} P_{j,j-k} j!\binom{r}{j} \right) Q_{r-k}^{'}(x)}{\sum_{k=0}^{q+1} k!\binom{r}{k} P_{k,k}} \tag{3.34}$$

Thus it is true for $m = q + 1$

We again fix n at $n = l$ and apply the principle of mathematical induction on m. With $n = l$, (3.8) becomes

$$Q_r^{(l)}(x) = \frac{l!\binom{r}{1} x^{r-l} - \sum_{k=1}^{m} \sum_{j=k}^{m} \left(j!\binom{r}{j} P_{j,j-k} \right) Q_{r-k}^{(l)}(x)}{\sum_{k=0}^{m} k!\binom{r}{k} P_{k,k}} \tag{3.35}$$

We try for $m = 1$,

$$Q_r^{(l)}(x) = \frac{l!\binom{r}{1} x^{r-l} - 1!\binom{r}{1} P_{1,0} Q_{r-1}^{(l)}(x)}{0!\binom{r}{0} P_{0,0} + 1!\binom{r}{1} P_{1,1}} \tag{3.36}$$

$$Q_r^{(l)}(x) = \frac{l!\binom{r}{1} x^{r-l} - rP_{1,0} Q_{r-1}^{(l)}(x)}{P_{0,0} + rP_{1,1}} \tag{3.37}$$

We assume that it is true for $m = q$, so that (3.35) becomes

$$Q_r^{(l)}(x) = \frac{l!\binom{r}{1} x^{r-l} - \sum_{k=1}^{q} \sum_{j=k}^{q} \left(j!\binom{r}{j} P_{j,j-k} \right) Q_{r-k}^{(l)}(x)}{\sum_{k=0}^{q} k!\binom{r}{k} P_{k,k}} \tag{3.38}$$

We now prove for $m = q + 1$.

From our construction of $Q_r^{(l)}(x)$ in (3.35) for $m = q$ up to $m = q + 1$, we have

$$Q_r^{(l)}(x) = \frac{l!\binom{r}{1}x^{r-l} - \left\{\sum_{k=1}^{q}\sum_{j=k}^{q}(P_{j,j-k})\, j!\binom{r}{j}\right\} Q_{r-k}^{(l)}(x)}{\sum_{k=0}^{q} k!\binom{r}{k}P_{k,k} + (q+1)!\binom{r}{q+1}P_{q+1,q+1}} + \frac{\left\{(P_{q+1,q-k+1})\,(q+1)!\binom{r}{q+1}\right\} Q_{r-k}^{(l)}(x)}{\sum_{k=0}^{q} k!\binom{r}{k}P_{k,k} + (q+1)!\binom{r}{q+1}P_{q+1,q+1}} \tag{3.39}$$

$$Q_r^{(l)}(x) = \frac{l!\binom{r}{1}x^{r-l}}{\sum_{k=0}^{q+1}(P_{k,k})\, k!\binom{r}{k}} - \frac{\sum_{k=1}^{q}\sum_{j=k}^{q}\left\{j!\binom{r}{j}(P_{j,j-k}) + (q+1)!\binom{r}{q+1}(P_{q+1,q+1})\right\} Q_{r-k}^{(l)}(x)}{\sum_{k=0}^{q+1}(P_{k,k})\, k!\binom{r}{k}} \tag{3.40}$$

$$Q_r^{(l)}(x) = \frac{l!\binom{r}{1}x^{r-l} - \sum_{k=1}^{q+1}\left(\sum_{j=k}^{q+1} j!\binom{r}{j}P_{j,j-k}\right) Q_{r-k}^{(l)}(x)}{\sum_{k=0}^{q} k!\binom{r}{k}P_{k,k}} \tag{3.41}$$

Thus it is true for $m = q + 1$ also.
Finally, we shall fix $n = l + 1$ and apply the principle of mathematical induction on m again.
With $n = l + 1$, (3.8) can now be written as

$$Q_r^{(l+1)}(x) = \frac{(l+1)!\binom{r}{r+1}x^{r-(l+1)} - \sum_{k=1}^{m}\sum_{j=k}^{m}\left(j!\binom{r}{j}P_{j,j-k}\right) Q_{r-k}^{(l+1)}(x)}{\sum_{k=0}^{m} k!\binom{r}{k}P_{k,k}} \tag{3.42}$$

Let us try for $m = 1$,

$$Q_r^{(l+1)}(x) = \frac{(l+1)!\binom{r}{r+1}x^{r-(l+1)} - 1!\binom{r}{1}P_{1,0}Q_{r-1}^{(l+1)}(x)}{0!\binom{r}{0}P_{0,0} + 1!\binom{r}{1}P_{1,1}} \tag{3.43}$$

$$Q_r^{(l+1)}(x) = \frac{(l+1)!\binom{r}{l+1}x^{r-(l+1)} - rP_{1,0}Q_{r-1}^{(l+1)}(x)}{P_{0,0} + rP_{1,1}} \tag{3.44}$$

And (3.44) is the same as (3.4) with n replaced by $l + 1$, thus it is true for $m = 1$.

Next, we assume that it is true for $m = q$, in which case, (3.42) can now be written as

$$Q_r^{(l+1)}(x) = \frac{(l+1)!\binom{r}{r+1}x^{r-(l+1)} - \sum_{k=1}^{q}\sum_{j=k}^{q}\left(j!\binom{r}{j}P_{j,j-k}\right)Q_{r-k}^{(l+1)}(x)}{\sum_{k=0}^{q}k!\binom{r}{k}P_{k,k}} \tag{3.45}$$

Now we want to prove that (3.42) holds for $m = q+1$.

From our construction of $Q_r^{(l+1)}(x)$ in (3.42) for $m = q$ up to $m = q+1$, we have

$$\begin{aligned} Q_r^{(l+1)}(x) &= \frac{(l+1)!\binom{r}{l+1}x^{r-(l+1)} - \left\{\sum_{k=1}^{q}\sum_{j=k}^{q}\left(P_{j,j-k}\right)j!\binom{r}{j}\right\}Q_{r-k}^{(l+1)}(x)}{\sum_{k=0}^{q}k!\binom{r}{k}P_{k,k} + (q+1)!\binom{r}{q+1}P_{q+1,q+1}} \\ &+ \frac{\left\{\left(P_{q+1,q-k+1}\right)(q+1)!\binom{r}{q+1}\right\}Q_{r-k}^{(l+1)}(x)}{\sum_{k=0}^{q}k!\binom{r}{k}P_{k,k} + (q+1)!\binom{r}{q+1}P_{q+1,q+1}} \end{aligned} \tag{3.46}$$

$$\begin{aligned} Q_r^{(l+1)}(x) &= \frac{(l+1)!\binom{r}{r+1}x^{r-(l+1)}}{\sum_{k=0}^{q+1}\left(P_{k,k}\right)k!\binom{r}{k}} \\ &- \frac{\sum_{k=1}^{q}\sum_{j=k}^{q}\left\{j!\binom{r}{j}\left(P_{j,j-k}\right) + (q+1)!\binom{r}{q+1}\left(P_{q+1,q+1}\right)\right\}Q_{r-k}^{(l)}(x)}{\sum_{k=0}^{q+1}\left(P_{k,k}\right)k!\binom{r}{k}} \end{aligned} \tag{3.47}$$

$$Q_r^{(l+1)}(x) = \frac{(l+1)!\binom{r}{l+1}x^{r-(l+1)} - \sum_{k=1}^{q+1}\left(\sum_{j=k}^{q+1}j!\binom{r}{j}P_{j,j-k}\right)Q_{r-k}^{(l+1)}(x)}{\sum_{k=0}^{q}k!\binom{r}{k}P_{k,k}} \tag{3.48}$$

From the foregoing, it can be concluded that (3.8) holds for all values of m, r and n. Thus, the validity of our formula.

CONCLUSION

The derivation of a general formula for the canonical polynomials associated with m-th order non-overdetermined linear ODE together with its associated n-th order derivative has been presented.

The formula are recursive and hence makes for easy determination of particular cases for which m will be specified. The use of canonical polynomial in the Tau approximation problem to the solution of ODEs is very attractive as they do not depend on the boundary conditions and when Tau approximations of higher degrees are needed, the process of their (i.e canonical polynomial) determination does not begin from the scratch. These are some of the shortcomings of the two other variants of the Tau method namely, the differential form and the integrated form.

It is intended that the polynomial reported above will be incorporated into the Tau method for purpose of generalizing the recursive formulation of the Tau method itself.

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