A General Method using Geometry to Find Eigen Vectors and Eigen Values of Matrix of Size 3x3

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Abstract—In this paper, the general model symmetric matrix 3x3 is expressed. Furthermore, the general model of non symmetric matrix 3x3 is discussed. The examples are presented to verify the results.

Keywords—Lagrangian multiplier; Cardan’s method; eigen value; eigen vector.

I. INTRODUCTION

Eigen values are greatest importance in dynamic problems (Luenberger 1979), (Johansen 1988), (Haftka and Adelman 1989) and many engineering application (Thomson 1996). The eigenvectors (Joy 2000) denoted by \( \lambda_1 \) and eigenvalues \( \lambda_i \) of a any matrix A that is satisfied \( Ax = \lambda x \). If A is an \( n \times n \) matrix, then \( x \) is an \( n \times 1 \) vector, and \( \lambda_i \) is a constant.

The matrix A has Eigen vectors and Eigen values are written as:

\[
V = \begin{pmatrix} x_1 & \cdots & x_n \\ \vdots & \ddots & \vdots \\ x_1 & \cdots & x_n \end{pmatrix}, D = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}
\]

The matrix V is called the modal matrix of A. Since D, as a diagonal matrix, has Eigen values \( \lambda_1, \ldots, \lambda_n \) which are the same as those of A then the matrices D and A are said to be similar. The transformation of A into D using \( V^{-1}A V = D \) is said to be a similarity transformation.

II. MATERIAL AND METHODS

- To find the model matrix and the Eigen values of a symmetric matrix:

\[
\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}
\]

We express the given matrix to demonstrate a surface of second degree

\[
\begin{pmatrix} a & h & g \\ x & y & z \end{pmatrix} \begin{pmatrix} x \\ h & b & f \\ g & f & c \end{pmatrix} = 0
\]

Then

\[
ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gxyz + 1 = 0
\]

With center \((0,0,0)\), i.e \( x_3 = y_3 = z_3 = 0 \) We take a function: \((x - x_3)^2 + (y - y_3)^2 + (z - z_3)^2 \) and we find its extreme points on the given surface, using Lagrange multiplier \( \lambda \). Then

\[
\phi(x, y, z) = (x - 0)^2 + (y - 0)^2 + (z - 0)^2 + \lambda(ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gxyz + l)
\]

We find:

\[
-\lambda = \frac{x}{ax + hy + gz} = \frac{y}{hx + by + fz} = \frac{z}{gx + fy + cz}
\]

and \( ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gxyz + l = 0 \)

We can find:

\[
h(x^2 - y^2) + (b - a)xy + fxz - gyz = 0, \\
g(x^2 - z^2) + (c - a)zx + fxy - hzy = 0, \\
f(y^2 - z^2) + (c - b)yz + gxy - hxz = 0
\]

From equation (4), we get:

\[
\frac{x}{hx - gy} = \frac{(x^2 - y^2) + (c - b)yz}{hx - gy}
\]

Multiply equation (2) by \( gf \). Multiply equation (3) by \(-fh\). Multiply equation (4) by \( gh \) and adding, then

\[
\begin{pmatrix} gb - fa + f^2h + g^2h \end{pmatrix}y + \\
\begin{pmatrix} gf^2 - fhc + fha - gh^2 \end{pmatrix}z
\]

\[-g^2f + fh^2 + cgh - bgh]yz = 0
\]

From equation (6), we get:

\[
\frac{x}{[g^2f - fh^2 - cgh + bgh]yz} = \frac{x}{[gb - fha - f^2h + g^2h]yz + [f^2h - fhc + fha - gh^2]z}
\]

From equations (5) and (7):

\[
f(y^2 - z^2) + (c - b)yz
\]

\[
hz - gy
\]

\[
[g^2f - fh^2 - cgh + bgh]yz
\]

The cubic equation in \( z \) is:

\[
-2(f^2h - fhc + fha - gh^2)z^3 + [-2(fgb - fha - f^2h + g^2h) + (c - b)(g^2f - fhc + fha - gh^2) - h(g^2f - fh^2 - cgh + bgh)]z^2y + [f(g^2f - fh^2 + fha - gh^2) +
\]
(c - b)(gfb - gfa - f^2h + g^2h) + g(g^2f - fh^2 - cgh + bhg)y^2z + [(gfb - gfa - f^2h + g^2h)]y^3 = 0
(8)

A_0 = -f(gf^2 - fhc + fha - gh^2)
A_1 = [-f(gfa - f^2h + gh^2) + (c - b)(gf^2 - fhc + fha - gh^2) - h(g^2f - fh^2 - cgh + bhg)y]
A_2 = [f(gfa - f^2h + gh^2) + (c - b)(gfb - gfa - f^2h + g^2h) + g(g^2f - fh^2 - cgh + bhg)y^2]
A_3 = [f(gfb - gfa - f^2h + g^2h)y^3]

Then the cubic equation (8) can be written in its reduced form:
N = A

For Cardan's method:

N = A_2 + 2kA_3 + 3k^2A_0
r = SQR[N^3 + ABS(M^3 + N^3)]
L = -N

if L < 0, then θ_0 = 180 - arccos(-L) = arccos(L),
if L > 0, then θ_0 = arccos(L),

θ_1 = \frac{\pi}{3}
R = 2\sqrt[3]{r}
z_1 = R \cos{θ_1} + k,
z_2 = R \cos{(θ_1 + 120)} + k,
z_3 = R \cos{(θ_1 + 240)} + k.

By the following relation

x_i = \frac{f(y_i^2 - z_i^2) + (c - b)y_i}{hzi - gy_i} ; i = 1,2,3

To find x_1, x_2, x_3 using z_1, z_2, z_3

The model matrix is

y_1 y_2 y_3
z_1 z_2 z_3

To find the Eigen values:

[a - λ h g] [y_1 y_2 y_3] = 0
[z_1 z_2 z_3] 0
Then

λ_1 = \frac{ax_1 + hy_1 + gz_1}{x_1}
λ_2 = \frac{ax_2 + hy_2 + gz_2}{x_2}
λ_3 = \frac{ax_3 + hy_3 + gz_3}{x_3}

Example (1):
Find the model matrix and Eigen values of the matrix

\begin{bmatrix}
11 & -6 & 2 \\
-6 & 10 & -4 \\
2 & -4 & 6 \\
\end{bmatrix}

Solution:

a = 11, b = 10, c = 6, h = -6, g = 2, f = -4 ≠ 0,

Then

A_0 = 320, A_1 = 480y, A_2 = -480y^2, A_3 = -320y^3, k = -0.5y,
M = \frac{-320y^3 + 0.5 + 480y^2 + 3 (0.5 y^2)^2}{3 * 320} = 0.

N = \frac{A_2 + 2kA_3 + 3k^2A_0}{3A_0} = \frac{-480y^2 + 2 * -0.5 * 480y^2 + 3 * (-0.5)^2 * 320y^2}{3 * 320} = -0.750y^2

r = 0.6495 y^3, L = 0, θ_0 = arccos L = \frac{π}{2}, θ_1 = \frac{π}{3}, R = \frac{2\sqrt{r}}{x} = 1.7321 y.

Then

z_1 = y_1, z_2 = -2 y_2, z_3 = -0.5 y_3,

x_1 = \frac{f(y_1^2 - z_1^2) + (c - b)y_1z_1}{h_{z_1} - gy_1} = 0.5 y_1,

x_2 = \frac{f(y_2^2 - z_2^2) + (c - b)y_2z_2}{h_{z_2} - gy_2} = 2y_2,

x_3 = \frac{f(y_3^2 - z_3^2) + (c - b)y_3z_3}{h_{z_3} - gy_3} = -y_3.

Then the model matrix is:

\begin{bmatrix}
x_1 & x_2 & x_3 \\
y_1 & y_2 & y_3 \\
z_1 & z_2 & z_3 \\
\end{bmatrix} = \begin{bmatrix}
0.5 & 2 & -1 \\
1 & 1 & 1 \\
-2 & -2 & 0.5 \\
\end{bmatrix}

λ_1 = \frac{ax_1 + hy_1 + gz_1}{x_1} = 3,

λ_2 = \frac{ax_2 + hy_2 + gz_2}{x_2} = 6,

λ_3 = \frac{ax_3 + hy_3 + gz_3}{x_3} = 18.

To check the results:

- By using Maple program
  > with(LinearAlgebra) :
  > E := \langle [-1, -6, 2], [-6, 10, -4], [-2, -4, 6] \rangle
  > \begin{bmatrix}
  11 & -6 & 2 \\
  -6 & 10 & -4 \\
  2 & -4 & 6 \\
  \end{bmatrix}
  \lambda_1 = \frac{ax_1 + hy_1 + gz_1}{x_1} = -1 \ \frac{2}{2} \ \frac{1}{2}
  \lambda_2 = \frac{ax_2 + hy_2 + gz_2}{x_2} = -1 \ \frac{2}{2} \ \frac{1}{2}
  \lambda_3 = \frac{ax_3 + hy_3 + gz_3}{x_3} = -1 \ \frac{2}{2} \ \frac{1}{2}

- By using Matlab program

>> A=[11 -6 2; -6 10 -4; 2 -4 6]

>> [V,D]=eig(A)
From equations (5) and (6):

\[ a = x \]

From equation (6), we get:

\[ b = \frac{x}{2} \]

From equation (4), we get:

\[ c = \frac{x}{3} \]

We take a function:

\[ \lambda = 2x \]

We express the given matrix to demonstrate a surface of second degree:

\[ Q = [a b c] [x y z] \]

Then:

\[ Q = ax^2 + by^2 + cz^2 + (d + f)xy + (k + h)yz + (e + g)xz + 1 = 0 \]

With center \((0,0,0)\), i.e \(x_s = y_s = z_s = 0\)

We find its extreme points on the given surface, using Lagrange multiplier \(\lambda\). Then

\[ \varphi(x,y,z) = x^2 + y^2 + z^2 + \lambda(ax^2 + by^2 + cz^2 + (d + f)xy + (k + h)yz + (e + g)xz + 1) \]

We find:

\[ \frac{\partial \varphi}{\partial x} = 0, \frac{\partial \varphi}{\partial y} = 0, \frac{\partial \varphi}{\partial z} = 0, \frac{\partial \varphi}{\partial \lambda} = 0, \text{ then:} \]

\[ -\lambda = \frac{2ax + (d + fy) + (e + gxz)}{2x} \]

\[ \frac{d + fy}{2x} = \frac{2y}{2x} \quad \text{and} \quad ax^2 + by^2 + cz^2 + (d + f)xy + (k + h)yz + (e + g)xz + 1 = 0 \]

We can find:

\[ (d + f)(x^2 - y^2) + 2(b - a)xy + (k + h)yz - (e + g)yz = 0, \]

\[ (e + g)(x^2 - z^2) + 2(c - a)xy + (k + h)xy - (d + f)yz = 0, \]

\[ (k + h)(y^2 - z^2) + 2(c - b)yz + (e + g)xy - (d + f)xz = 0, \]

From equation (4), we get:

\[ x = \frac{(k + h)(y^2 - z^2) + 2(c - b)yz}{(d + f)z - (e + g)y} \]

Multiply equation (2) by \((e + g)(k + h)\), multiply equation (3) by \(-(k + h)(d + f)\), multiply equation (4) by \((e + g)(d + f)\) and adding, then:

\[ 2(b - a)(e + g)(k + h) - (d + f)(k + h)^2 + (d + f)(e + g)^2 \]

\[ [(e + g)(k + h)^2 - 2(c - a)(d + f)(k + h) - (e + g)(d + f)^2]x + [(k + h)(e + g)^2 + (k + h)(d + f)^2 + 2(c - b)(e + g)(d + f)]y = 0 \]

From equation (6), we get:

\[ x = \frac{1}{2} \left( k + h \right) \]

\[ y = \frac{1}{2} \left( e + g \right) \]

\[ z = \frac{1}{2} \left( a + d + f \right) \]

\[ \lambda = \frac{1}{2} \left( a + d + f \right) \]

Then the cubic equation (8) can be written in its reduced form:

\[ A_0z^3 + (kA_1 + A_2)z + (A_3 + kA_2 + \frac{4}{3}k^2A_1) = 0 \]

Using Cardan's method we find the three roots of equation (9):

\[ z_1', z_2', z_3' \]

We obtain the three roots of equation (8):

\[ z_1 = z_1 + k, \quad z_2 = z_2 + k, \quad z_3 = z_3 + k \]

By the following relation:

\[ z = \frac{(k + h)(y^2 - z^2) + 2(c - b)yz}{(d + f)z - (e + g)y} \]

To find \(x_1, x_2, x_3\) using \(z_1, z_2, z_3\):

\[ \begin{align*}
  \lambda_1 &= \frac{1}{2} \left( a + d + f \right) \left( y_1 - \frac{e + g}{2} \right) z_1 = 1,2,3
\end{align*} \]

To find the Eigen values:

\[ \lambda_1 = \frac{1}{2} \left( a + d + f \right) \left( y_1 - \frac{e + g}{2} \right) z_1 = 1,2,3 \]

From equations (5) and (6):
Example(2):

Find the model matrix and the Eigen values of non symmetric matrix

\[
A = \begin{bmatrix}
11 & -4 & 1 \\
-8 & 10 & -6 \\
3 & -2 & 6
\end{bmatrix}
\]

Solution:

\[
a = 11, b = 10, c = 6, d = -4, e = 1, f = -8, k = -6, g = 3, h = -2
\]

Since

\[
A = \begin{bmatrix}
a & d & e \\
f & b & k \\
g & h & c
\end{bmatrix} \text{ is equivalent to the symmetric matrix }
\begin{bmatrix}
a & h & g \\
g & f & k \\
h & k & c
\end{bmatrix}
\]

Where \( h = \frac{f + d}{2}, g = \frac{e + g}{2}, f = \frac{k + h}{2} \).

The non symmetric matrix

\[
\begin{bmatrix}
a & d & e \\
f & b & k
\end{bmatrix}
\]

The examples is discussed in the above sections provide us a way to generalize the model matrix for non symmetric matrix.

REFERENCES