A General Method using Geometry to Find Eigen Vectors and Eigen Values of Matrix of Size 3x3

Rania. B. M. Amer
Department of Engineering Mathematics and Physics,
Faculty of Engineering, Zagazig University, P.O. 44519,
Zagazig, Egypt

M. Afwat
Department of Engineering Mathematics and Physics,
Faculty of Engineering, Zagazig University, P.O. 44519,
Zagazig, Egypt

***Abstract*— In this paper, the general model symmetric matrix 3x3 is expressed. Furthermore, the general model of non symmetric matrix 3x3 is discussed. The examples are presented to verify the results.**

***Keywords*— Lagrange multiplier ; Cardan's method ; eigen value; eigen vector.**

I. INTRODUCTION

Eigen values are greatest importance in dynamic problems(Luenberger 1979), (Johansen 1988),(Haftka and Adelman 1989) and many engineering application (Thomson 1996). The eigenvectors (Joy 2000) denoted by x_i and eigenvalues λ_i of a any matrix A that is satisfied Ax = λx. If A is an n x n matrix, then x_i is an n x 1 vector, and λ_i is a constant.

The matrix A has Eigen vectors and Eigen values are written as:

$$V=\begin{pmatrix} x_1^1 & \cdots & x_1^n \\ \vdots & \ddots & \vdots \\ x_n^1 & \cdots & x_n^n \end{pmatrix}, D=\begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

The matrix V is called the modal matrix of A. Since D, as a diagonal matrix, has Eigen values $\lambda_1, \ldots\ldots, \lambda_n$ which are the same as those of A then the matrices D and A are said to be similar. The transformation of A into D using V −1A V = D is said to be a similarity transformation.

II. MATERIAL AND METHODS

- **To find the model matrix and the Eigen values of a symmetric matrix:**

$$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

We express the given matrix to demonstrate a surface of second degree

$$[x \quad y \quad z]\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Then

$$ax^2+by^2+cz^2+2hxy+2fyz+2gxz+l=0 \qquad (1)$$

With center(0,0,0), i.e $x_s=y_s=z_s=0$

We take a function: $(x-x_s)^2+(y-y_s)^2+(z-z_s)^2$ and we find its extreme points on the given surface, using Lagrange multiplier λ. Then

$$\varphi(x,y,z)=(x-0)^2+(y-0)^2+(z-0)^2+\lambda(ax^2+by^2+cz^2+2hxy+2fyz+2gxz+l)$$

We find:

$$\frac{\partial\varphi}{\partial x}=0,\ \frac{\partial\varphi}{\partial y}=0,\ \frac{\partial\varphi}{\partial z}=0\ ,\ \frac{\partial\varphi}{\partial\lambda}=0,\ \text{then:}$$

$$-\lambda=\frac{x}{ax+hy+gz}=\frac{y}{hx+by+fz}=\frac{z}{gx+fy+cz} \quad \text{and} \quad ax^2+by^2+cz^2+2hxy+2fyz+2gxz+l=0$$

We can find:

$$h(x^2-y^2)+(b-a)xy+fxz-gyz=0, \qquad (2)$$

$$g(x^2-z^2)+(c-a)xz+fxy-hyz=0, \qquad (3)$$

$$f(y^2-z^2)+(c-b)yz+gxy-hxz=0, \qquad (4)$$

From equation (4), we get:

$$x=\frac{f(y^2-z^2)+(c-b)yz}{hz-gy} \qquad (5)$$

Multiply equation (2) by gf, Multiply equation (3) by −fh, Multiply equation (4) by gh and adding, then

$$[gfb-gfa-f^2h+g^2h]xy+[gf^2-fhc+fha-gh^2]xz+[-g^2f+fh^2+cgh-bgh]yz=0 \qquad (6)$$

From equation (6), we get:

$$x=\frac{[g^2f-fh^2-cgh+bgh]yz}{[gfb-gfa-f^2h+g^2h]y+[f^2-fhc+fha-gh^2]z} \qquad (7)$$

From equations (5) and (7):

$$\frac{f(y^2-z^2)+(c-b)yz}{hz-gy}=\frac{[g^2f-fh^2-cgh+bgh]yz}{[gfb-gfa-f^2h+g^2h]y+[gf^2-fhc+fha-gh^2]z}$$

The cubic equation in z is:

$$[-f(gf^2-fhc+fha-gh^2)]z^3+[-f(gfb-gfa-f^2h+g^2h)+(c-b)(gf^2-fhc+fha-gh^2)-h(g^2f-fh^2-cgh+bgh)]z^2y+[f(gf^2-fhc+fha-gh^2)+$$

$(c-b)(gfb-gfa-f^2h+g^2h)+g(g^2f-fh^2-cgh+bgh)]y^2z+[f(gfb-gfa-f^2h+g^2h)]y^3=0$
(8)

$$A_0=-f(gf^2-fhc+fha-gh^2)$$
$$A_1=[-f(gfb-gfa-f^2h+g^2h)+(c-b)(gf^2-fhc+fha-gh^2)-h(g^2f-fh^2-cgh+bgh)]y$$
$$A_2=[f(gf^2-fhc+fha-gh^2)+(c-b)(gfb-gfa-f^2h+g^2h)+g(g^2f-fh^2-cgh+bgh)]y^2$$
$$A_3=[f(gfb-gfa-f^2h+g^2h)]y^3$$
$$k=\frac{-A_1}{3A_0}$$

Then the cubic equation (8) can be written in its reduced form:

$$A_0z^3+(A_2+kA_1)z+\left(A_3+kA_2+\tfrac{2}{3}k^2A_1\right)=0 \qquad (9)$$

For Cardan's method:

$$M=\frac{A_3+kA_2+k^2A_1+k^3A_0}{2A_0}=\frac{A_3+kA_2+\frac{2}{3}k^2A_1}{2A_0},$$
$$N=\frac{A_2+2kA_1+3k^2A_0}{3A_0}=\frac{A_2+kA_1}{3A_0}$$
$$r=SQR[M^2+ABS(M^3+N^3)],$$
$$L=\frac{-M}{r}$$

if $L<0$, then $\theta_0=180-\arccos(-L)=\arccos(L)$,

if $L>0$, then $\theta_0=\arccos(L)$,

$$\theta_1=\frac{\theta_0}{3},$$
$$R=2\sqrt[3]{r}$$
$$z_1=R\ \cos\theta_1+k,$$
$$z_2=R\ \cos(\theta_1+120)+k,$$
$$z_3=R\ \cos(\theta_1+240)+k.$$

By the following relation

$$x_i=\frac{f(y_i^2-z_i^2)+(c-b)y_iz_i}{hz_i-gy_i};i=1,2,3$$

To find x_1,x_2,x_3 using z_1,z,z_3

The model matrix is $\begin{bmatrix}x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\\ z_1 & z_2 & z_3\end{bmatrix}$

To find the Eigen values:

$$[a-\lambda \quad h \quad g]\begin{bmatrix}x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\\ z_1 & z_2 & z_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Then

$$\lambda_1=\frac{ax_1+hy_1+gz_1}{x_1},$$
$$\lambda_2=\frac{ax_2+hy_2+gz_2}{x_2},$$
$$\lambda_3=\frac{ax_3+hy_3+gz_3}{x_3}$$

Example (1):

Find the model matrix and Eigen values of the matrix $\begin{bmatrix}11 & -6 & 2\\ -6 & 10 & -4\\ 2 & -4 & 6\end{bmatrix}$

Solution:

$$a=11,b=10,c=6,h=-6,g=2,f=-4\neq 0,$$

Then

$A_0=320,\ A_1=480y,\ A_2=-480y^2,\ A_3=-320y^3,k=-0.5y,$

$$M=\frac{-320y^3+0.5*480y^3+\frac{2}{3}(0.5)^2480y^3}{2*320}=0,$$
$$N=\frac{A_2+2kA_1+3k^2A_0}{3A_0}=\frac{-480y^2+2*-0.5*480y^2+3*(-0.5)^2*320y^2}{3*320}=-0.7500y^2$$

$r=0.6495\ y^3, L=0, \theta_0=\arc\cos L=\frac{\pi}{2},\theta_1=\frac{\theta_0}{3}=\frac{\pi}{6},R=2\sqrt[3]{r}=1.7321\ y$

Then

$$z_1=y_1,z_2=-2\,y_2,z_3=-0.5\,y_3,$$
$$x_1=\frac{f(y_1^2-z_1^2)+(c-b)y_1z_1}{hz_1-gy_1}=0.5\,y_1,$$
$$x_2=\frac{f(y_2^2-z_2^2)+(c-b)y_2z_2}{hz_2-gy_2}=2y_2,$$
$$x_3=\frac{f(y_3^2-z_3^2)+(c-b)y_3z_3}{hz_3-gy_3}=-y_3,$$

Then the model matrix is:

$$\begin{bmatrix}x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\\ z_1 & z_2 & z_3\end{bmatrix}=\begin{bmatrix}0.5 & 2 & -1\\ 1 & 1 & 1\\ 1 & -2 & -0.5\end{bmatrix}$$

$$\lambda_1=\frac{ax_1+hy_1+gz_1}{x_1}=3,$$
$$\lambda_2=\frac{ax_2+hy_2+gz_2}{x_2}=6,$$
$$\lambda_3=\frac{ax_3+hy_3+gz_3}{x_3}=18$$

To check the results:

- By using Maple program

> *with*(*LinearAlgebra*) :

$E := \langle\langle 11, -6, 2\rangle|\langle -6, 10, -4\rangle|\langle 2, -4, 6\rangle\rangle$

$$\begin{bmatrix}11 & -6 & 2\\ -6 & 10 & -4\\ 2 & -4 & 6\end{bmatrix}$$

Eigenvalues(*E*, *output* = '*list*')

[3, 6, 18]

v, *e* := *Eigenvectors*(*E*)

$$\begin{bmatrix}6\\18\\3\end{bmatrix},\begin{bmatrix}-1 & 2 & \frac{1}{2}\\ -\frac{1}{2} & -2 & 1\\ 1 & 1 & 1\end{bmatrix}$$

- By using Matlab program

```
>> A=[11 -6 2;-6 10 -4;2 -4 6]

>> [V,D] = eig(A)
```

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V =

0.3333	-0.6667	0.6667
0.6667	-0.3333	-0.6667
0.6667	0.6667	0.3333

D =

3.0000	0	0
0	6.0000	0
0	0	18.0000

- **To find the model matrix and the Eigen values of a given non symmetric matrix:**

$$A = \begin{bmatrix} a & d & e \\ f & b & k \\ g & h & c \end{bmatrix}$$

We express the given matrix to demonstrate a surface of second degree

$$Q = [x \quad y \quad z] \begin{bmatrix} a & d & e \\ f & b & k \\ g & h & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Then

$$Q = ax^2 + by^2 + cz^2 + (d+f)xy + (k+h)yz + (e+g)xz + l = 0 \qquad (10)$$

With center(0,0,0), i.e $x_s = y_s = z_s = 0$

We take a function: $(x - x_s)^2 + (y - y_s)^2 + (z - z_s)^2$ and we find its extreme points on the given surface, using Lagrange multiplier λ. Then

$$\varphi(x, y, z) = x^2 + y^2 + z^2 + \lambda(ax^2 + by^2 + cz^2 + (d+f)xy + (k+h)yz + (e+g)xz + l)$$

We find:

$$\frac{\partial\varphi}{\partial x} = 0,\ \frac{\partial\varphi}{\partial y} = 0,\ \frac{\partial\varphi}{\partial z} = 0,\ \frac{\partial\varphi}{\partial\lambda} = 0,\ \text{then:}$$

$$-\lambda = \frac{2x}{2ax + (d+f)y + (e+g)z} = \frac{2y}{(d+f)x + 2by + (k+h)z} = \frac{2z}{(e+g)x + (k+h)y + 2cz} \quad \text{and} \quad ax^2 + by^2 + cz^2 + (d+f)xy + (k+h)yz + (e+g)xz + l = 0$$

We can find:

$$(d+f)(x^2 - y^2) + 2(b-a)xy + (k+h)xz - (e+g)yz = 0, \qquad (2)$$

$$(e+g)(x^2 - z^2) + 2(c-a)xz + (k+h)xy - (d+f)yz = 0, \qquad (3)$$

$$(k+h)(y^2 - z^2) + 2(c-b)yz + (e+g)xy - (d+f)xz = 0, \qquad (4)$$

From equation (4), we get:

$$x = \frac{(k+h)(y^2 - z^2) + 2(c-b)yz}{(d+f)z - (e+g)y} \qquad (5)$$

Multiply equation (2) by $(e+g)(k+h)$, multiply equation (3) by $-(k+h)(d+f)$, multiply equation (4) by $(e+g)(d+f)$ and adding, then

$$[2(b-a)(e+g)(k+h) - (d+f)(h+k)^2 + (d+f)(e+g)^2] [(e+g)(k+h)^2 - 2(c-a)(d+f)(k+h) - (e+g)(d+f)^2]xz + [-(k+h)(e+g)^2 + (k+h)(d+f)^2 + 2(c-b)(e+g)(d+f)]yz = 0 \qquad (6)$$

From equation (6), we get:

$$x = \frac{[(k+h)(e+g)^2 - (k+h)(d+f)^2 - 2(c-b)(e+g)(d+f)]yz}{[2(b-a)(e+g)(k+h) - (d+f)(h+k)^2 + (d+f)(e+g)^2]y + [(e+g)(k+h)^2 - 2(c-a)(d+f)(k+h) - (e+g)(d+f)^2]\, z}$$

From equations (5) and (6):

$$\frac{(k+h)(y^2 - z^2) + 2(c-b)yz}{(d+f)z - (e+g)y} = \frac{[(k+h)(e+g)^2 - (k+h)(d+f)^2 - 2(c-b)(e+g)(d+f)]yz}{[2(b-a)(e+g)(k+h) - (d+f)(h+k)^2 + (d+f)(e+g)^2]y + [(e+g)(k+h)^2 - 2(c-a)(d+f)(k+h) - (e}$$

Then

$$A_0 = -(k+h)[(e+g)(k+h)^2 - 2(c-a)(d+f)(k+h) - (e+g)(d+f)^2]$$

$$A_1 = [-(k+h)[2(b-a)(e+g)(k+h) - (d+f)(h+k)^2 + (d+f)(e+g)^2] + 2(c-b)[(e+g)(k+h)^2 - 2(c-a)(d+f)(k+h) - (e+g)(d+f)^2] - (d+f)[(k+h)(e+g)^2 - (k+h)(d+f)^2 - 2(c-b)(e+g)(d+f)]]y$$

$$A_2 = [(k+h)[(e+g)(k+h)^2 - 2(c-a)(d+f)(k+h) - (e+g)(d+f)^2] + 2(c-b)[2(b-a)(e+g)(k+h) - (d+f)(h+k)^2 + (d+f)(e+g)^2] + (e+g)[(k+h)(e+g)^2 - (k+h)(d+f)^2 - 2(c-b)(e+g)(d+f)]]y^2$$

$$A_3 = [(k+h)[2(b-a)(e+g)(k+h) - (d+f)(h+k)^2 + (d+f)(e+g)^2]]y^3$$

$$k = \frac{-A_1}{3A_0}$$

Then the cubic equation (8) can be written in its reduced form:

$$A_0 z^3 + (kA_1 + A_2)z + \left(A_3 + kA_2 + \frac{2}{3}k^2 A_1\right)$$

Using Cardan's method we find the three roots of equation (9)

$$z_1', \ z_2', \ z_3'$$

We obtain the three roots of equation (8):

$$z_1 = z_1' + k, \qquad z_2 = z_2' + k, \qquad z_3 = z_3' + k,$$

By the following relation

$$x = \frac{(k+h)(y_i^2 - z_i^2) + 2(c-b)y_i z_i}{(d+f)z_i - (e+g)y_i};\ i = 1,2,3$$

To find x_1, x_2, x_3 using z_1, z, z_3

$$\text{The model matrix is } V = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

To find the Eigen values:

$$\begin{bmatrix} a - \lambda & (d+f)/2 & (e+g)/2 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} = [0 \quad 0 \quad 0]$$

Then

$$\lambda_i = \frac{ax_i + \left(\frac{d+f}{2}\right) y_i + \left(\frac{e+g}{2}\right) z_i}{x_i},\ i = 1,2,3$$

Example(2):

Find the model matrix and the Eigen values of non symmetric matrix

$$A = \begin{bmatrix} 11 & -4 & 1 \\ -8 & 10 & -6 \\ 3 & -2 & 6 \end{bmatrix}$$

Solution:

$$a = 11, b = 10, c = 6, d = -4, e = 1, f = -8, k = -6, g = 3, h = -2$$

Since

$$A_0 = 5120$$
$$A_1 = 7680\,y$$
$$A_2 = -7680\,y^2$$
$$A_3 = -5120y^3$$

$$k = \frac{-A_1}{3A_0} = -0.5000y$$

$$M = \frac{A_3 + kA_2 + k^2A_1 + k^3A_0}{2A_0} = \frac{A_3 + kA_2 + \frac{2}{3}k^2A_1}{2A_0} = 0 \,,$$

$$N = \frac{A_2 + 2kA_1 + 3k^2A_0}{3A_0} = \frac{A_2 + kA_1}{3A_0} = -0.7500\,y^2$$

$$r = SQR[M^2 + ABS(M^2 + N^3)] = 0.6495\,y^3$$

$$\Delta = M^2 + N^3 = -0.4219 < 0$$

Then the three roots are real

$$L = \frac{-M}{r} = 0$$

$$\text{if } L < 0, \text{then } \theta_0 = 180 - \arccos(-L) = \arccos(L),$$
$$\text{if } L > 0, \text{then } \theta_0 = \arccos(L) = 90\,,$$

$$\theta_1 = \frac{\theta_0}{3} = 30,$$

$$R = 2\sqrt[3]{r} = 1.7321y$$

$$z_1 = R\,\cos\theta_1 + k = y_1,$$
$$z_2 = R\,\cos(\theta_1 + 120) + k = -2\,y_2\,,$$
$$z_3 = R\,\cos(\theta_1 + 240) + k = -0.5\,y_3\,.$$

$$x_1 = \frac{(k+h)(y_1^2 - z_1^2) + 2(c-b)y_1z_1}{(d+f)z_1 - (e+g)y_1} = 0.5\,y_1$$
$$x_2 = 2y_2,$$
$$x_3 = -y_3$$

Then the model matrix is:

$$V = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} = \begin{bmatrix} 0.5 & 2 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & -0.5 \end{bmatrix},$$

$$\lambda_i = \frac{ax_i + (\frac{d+f}{2})\,y_i + (\frac{e+g}{2})z_i}{x_i}, i = 1,2,3$$

So

$$\lambda_1 = 3,$$
$$\lambda_2 = 6,$$
$$\lambda_3 = 18$$

III. CONCLUSION

The examples is discussed in the above sections provide us a way to generalize the model matrix for non symmetric matrix.

The non symmetric matrix $\begin{bmatrix} a & d & e \\ f & b & k \\ g & h & c \end{bmatrix}$ is equivelant to the symmetric matrix $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & k \end{bmatrix}$, Where $h = \frac{f+d}{2}, g = \frac{e+g}{2}, f = \frac{k+h}{2}$.

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