A Fixed Point Theorem on Semi-Metric Space using Occasionally Weakly Compatible Mappings

B. Vijayabasker Reddy
Department of Mathematics, Sreenidhi Institute of Science and Technology, Ghatkesar, Hyderabad, India -501 301

V. Srinivas
Department of Mathematics, University college of Science, Saifabad, Osmania University, Hyderabad, India.

Abstract- The aim of this paper is to prove common fixed point theorem for four self mappings in semi-metric space using the concept of occasionally weakly compatible. This theorem generalizes the result of Bijendra Singh and M.S Chauhan[1].

Keywords- Semi-metric space, coincidence point, weakly compatible, occasionally weakly compatible, Fixed point.

I. INTRODUCTION
The concept of semi-metric space is introduced by Menger, which is a generalization of metric space. Cicchese introduced the notion of a contractive mappings in semi-metric space and proved the fixed point theorem. In 2006 Jungck and Rhoades introduced the concept of Occasionally weakly compatible mappings which generalizes weakly compatible mappings.

II. PRELIMINARIES
Definition 1: (X, d) is said to be Semi-metric space if and only if it satisfies the following conditions:
M1: d(x, y)=0 if and only if x=y.
M2: d(x, y)=d(y, x) if and only if x=y
for any x, y ∈ X.

Definition 2: Let A and B be two self mappings of a semi-metric space (X, d) then A and B are said to be weakly compatible mappings if they commute at their coincidence points.

Definition 3: Let A and B be two self maps of a semi-metric space (X, d) then A and B are said to be occasionally weakly compatible mappings if there is a coincidence point x ∈ X of A and B at which A and B commute.

Remark 1: Weakly compatible mappings are occasionally compatible mappings but converse is not true.

Example 1: Let (X, d) be semi-metric space with X=[1/2, 5] and d(x, y)=(x-y)^2. Define two self mappings A and B as follows:
A(x) = \begin{cases} 
  x^2 & \text{if } 1 \leq x < 1 \\
  2x - 1 & \text{if } x \geq 1
\end{cases}
and
B(x) = \begin{cases} 
  2x & \text{if } 1 \leq x < 1 \\
  x^2 & \text{if } x \geq 1
\end{cases}

Clearly, X=1/2 and x=1 are two coincidence points. If x=1 then A(1)=1=B(1) which gives AB(1)=BA(1). If x=1/2 then A(1/2)=B(1/2)=1/4 but AB(1/2) ≠ BA(1/2). Therefore A and B are occasionally weakly compatible but not weakly compatible.

Lemma 1: Let (X, d) be a semi-metric space, A, B are occasionally weakly compatible mappings of X. If the self mappings A and B on X have a unique point of coincidence w=Ax=Bx. Then w is unique common fixed point of A and B.

Proof: Since A and B are occasionally weakly compatible mappings, there exists a point x ∈ X such that Ax=Bx=w and ABx=BAx. Thus AAx=ABx=BAx Which gives Ax is also point of coincidence of A and B. Since the point of coincidence w=Ax is unique then, BAx=AAx=Ax and w=Ax is a common fixed point of A and B. If z is any common fixed point of A and B then z=Az=Bz=w by the uniqueness of the point of coincidence.
III MAIN RESULT

Theorem 1: Let A,B,S,T,P and Q be self maps on a semi metric space \((X,d)\) If

(i) \((AP,S)\) and \((BQ,T)\) are occasionally weakly compatible mappings.

(ii) \[
\left[ d(APx, BQy) \right]^2 \leq k_1 \left[ d(APx, Sx) d(BQy, Ty) + d(BQy, Sx) d(APx, Ty) \right] + k_2 \left[ d(APx, Apx) d(APx, BQy) + d(BQy, BQy) d(APx, APx) \right]
\]

Where \(x,y \in X\) and \(k_1 + 2k_2 \leq 1, k, k_2 \geq 0\)

then \(AP, BQ, S\) and \(T\) have a Common fixed point. Further if \(AP=PA\) and \(BQ=QB\) then \(A, B, P, Q, S\) and \(T\) have a common fixed point.

Proof: \((AP, S)\) and \((BQ, T)\) are occasionally weakly compatible, then there exists some \(x,y \in X\) such that

\(APx=Sx\) and \(BQy=Ty\). Using (ii) we claim \(APx=BQy\).

\[
\left[ d(APx, BQy) \right]^2 \leq k_1 \left[ d(APx, Sx) d(BQy, Ty) + d(BQy, Sx) d(APx, Ty) \right] + k_2 \left[ d(APx, Apx) d(APx, BQy) + d(BQy, BQy) d(APx, APx) \right]
\]

This is contradiction. So \(APx=BQy\).

Therefore \(APx=BQy=Sx=Ty\).

if there is another point of coincident say, \(w\) such that \(APz=Sz=w\) then \(APz=Sz=Ty\). Which gives \(APz=APx\) implies \(z=x\).

Hence \(w=APx=Sx\) for \(w \in X\) is the unique point of coincidence of \(AP\) and \(S\). By lemma (1.1) \(w\) is a fixed point of \(AP\) and \(S\). Hence \(APw=Sw=w\). Similarly there exists a common fixed point \(u \in X\) such that \(u=BQu=Tu\).

Suppose \(u\neq w\)

Put \(x=w\) and \(y=u\) in (ii)

\[
[d(w,u)]^2 = [d(APw, BQu)]^2 \leq k_1 \left[ d(APw, Sw) d(BQu, Tu) + d(BQu, Sw) d(APw, Tu) \right] + k_2 \left[ d(APw, Sw) d(APw, Tu) + d(BQu, Sw) d(BQu, Tu) \right]
\]

This is contradiction. Therefore \(u=w\). Hence \(w\) is unique common fixed point of \(AP, BQ, S\) and \(T\).

If \(AP=PA\) and \(BQ=QB\) then \(Aw=A(Pw)=A(Paw)=AP(Aw)\).

Put \(x=w\) and \(y=Aw\) in (ii)

\[
[d(Aw, Aw)]^2 \leq k_1 \left[ d(Aw, Aw) d(w, Aw) \right] + k_2 \left[ d(Aw, Aw) d(Aw, Aw) + d(BQ(Aw), Sw) d(BQ(Aw), Sw) \right]
\]

Which gives \(w=Aw\)

\(Pw=A(Pw)=P(Aw)=w\).

Similarly we have \(Bw=Qw=w\).

Hence \(A, B, S, T, P\) and \(Q\) have unique fixed point.

Example 2:

Let \((X,d)\) be the semi metric Space with \(X=\{0,1/2\}\) and \(d = (x-y)^2\).

Define Self mappings \(A, B, T, S, P, Q\) as

\[
A(x) = \frac{2x+1}{4}, B(x) = \frac{4x+1}{6}, T(x) = \frac{4x+3}{10},
\]

\[
S(x) = \frac{6x+1}{8}, P(x) = \frac{2+6x}{10} \quad \text{and} \quad Q(x) = \frac{2x+3}{8}.
\]
Also the mappings satisfy all the conditions of theorem 1. Here the common fixed point is 1/2

REFERENCES: