A Discrete Time Inventory System With Retrial Demands

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Abstract

In this paper, a discrete time inventory system with demands occurring according to a Bernoulli process. The inventory is replenished according to an (s, S) policy and the lead times are assumed to follow a geometric distribution. The demands that occur during stock out period is permitted enter into the orbit of finite size. These orbiting demands retry to get their requirement after some random time and the time interval between two consecutive requests is distributed as geometric and the retrial policy is the multiplicative retrial policy. The joint probability distribution of the number of demands in the orbit and the inventory level is obtained in steady state case. Some system performance measures are derived and the results are illustrated numerically.

Keywords- Discrete time inventory system, (s, S) policy, Retrial demand, Multiplicative retrial policy

1. Introduction

Inventory models have been considered under continuous review as well as periodic review. In the recent past discrete time models have started receiving attention of researchers in the areas of queuing and telecommunications [2, 3]. In discrete time setting, it is assumed that the time axis is calibrated into epochs by small units and that all the events are deemed to occur only at these epochs. With the advent of fast computing devices and efficient transaction reporting facilities, such epochs with small gaps can be conveniently assumed so that events can occur at these epochs.

In the case of inventory modelling under discrete times, the first paper was by Bar-Lev and Perry [7], who assumed that demands are non-negative integer value random variables and items have constant life times. Lian and Liu [10] developed a discrete time inventory model with geometrically distributed inter-demand times, bulk demands and constant life time for items. They assumed (0, S) ordering policy, with instantaneous supply which clears any backlog and restores the stock to the maximum capacity S. This assumption helped them to have fixed life time for all items. They derived the limiting distribution of inventory level through matrix-analytic method.

Abboud [1] studied a discrete inventory model for production inventory systems with machine breakdowns. They assumed that the demand and production rates were constant and that the failure and repair times of each item were independently distributed as geometric. Lian et al. [11] developed a discrete time inventory system with discrete PH-renewal process for (batch) demand time points and assumed discrete- PH-distribution for life time of items. They also assumed zero lead time and that unsatisfied demand were completely backlogged.

I.Atentia and P.Moreno [6] developed a discrete time retrial queueing system with multiplicative repeated attempts. The authors used the retrial policy is the multiplicative one which analogous to the linear retrial policy in continuous-time [4].

The rest of the paper is organized as follows. In Section 2, we describe the problem under consideration. The mathematical model of our problem is presented in section 3. The limiting probability vector of the chain is calculated in Section 4. In section 5 we derived some important system performance measures and in the final section the expression for total expected cost rate and we provide few numerical illustrations of the results.

Notation:

- $[A]_{ij}$: entry at (i, j) th position of A
- 0 : zero vector of appropriate dimension
- I : identity matrix of appropriate dimension
- e : a column vector of 1's with appropriate dimension
- $\delta_{i,j}$: Kronecker delta function

2. Problem formulation

We consider a discrete time retrial inventory system where the time axis is divided into intervals of equal length, called slots (epochs). It is assumed that all system activities (arrivals, retrials and replenishment) occur at the slot boundaries, and therefore they may occur at the same time. The maximum capacity of the inventory is S.

- Demand arrive according to a Bernoulli process with probability a, thus a is the probability that a demand arrives at a slot and $\bar{a}$ $(= 1 - a)$, is the probability that an arrival does not take place in a slot. When the on-hand inventory level is more than one then the arriving demand is satisfied immediately.
- We adopt (s, S) ordering policy, that is, when the number of available items reaches the value s, an order for $Q(= S - s > s)$ items is placed which is delivered after a lead time of geometric distribution with parameter $c > 0$. The condition $Q > s$ is assumed so that when the supply of an order is received during the stock out period, the inventory level would be brought above the reorder level.
- If the arrival finds the inventory level zero, he enters into the orbit of pending demands and he retry after a random amount of time. In order to overcome analytical difficulties, we assume the most natural approach of restricting the retrial group to be of finite size. The arrivals of demands at the time of empty stock with full retrial orbit are assumed to be lost. So, in what follows, M will denote the maximum number of simultaneous pending demand in which the probability for the demand to be lost is negligible. If more than one repeated demand retries at the same slot, any of them is randomly selected and the others must go back to the retrial group. The time between two successive repeated attempts is geometrically distributed with probability $1 - r_1^{1-\delta_{0,k}} r_2^k (r_1, r_2 \in [0,1],\ r_1 r_2 \neq 1$ given that there are k demands in the orbit. If $r_2 = 1$, the retrial policy becomes the constant retrial policy, when $r_1 = 1$, we get the classical retrial policy.

Unlike continuous review inventory systems, multiple events such as demand, supply and retrial from the orbit may occur between epochs n and $n + 1, n = 0, 1, 2, \ldots$. Hence we adopt the following convention: If the events such as demand for an item, retrial from the orbit and supply of an order takes place at $n\ (n = 1, 2, 3. \ldots)$, it is assumed that first supply is received then demand occurs and finally retrial from the orbit takes place.

3. Model Descriptions

Let Xn and Ln denote respectively the number of demands in the orbit and the inventory level at time n. From the assumptions made on the input and output processes, it can be shown that the stochastic process $(X, L) = \{(X_n, L_n), n \in N\}$ is a Discrete Time Markov Chain with state space given by $E = \{(i, j) : i = 0, 1, \ldots, M, j = 0, 1, \ldots, S\}$.

The transition probability function is defined as,

$$p((i, j), (k, l)) = \Pr[X_{n+1} = k, L_{n+1} = l | X_n = i, L_n = j], (i, j), (k, l) \in E,$$

The transition probability matrix P of this process, $P = ((\, p((i, j), (k, l))\,)), (i, j), (k, l) \in E$ is given by,

$$[P]_{kl} = \begin{cases} A_0 & l = k + 1 \quad k = 0,1, \ldots, M - 1 \\ B_k & l = k \quad k = 0,1, \ldots, M \\ C_k & l = k - 1 \quad k = 1,2, \ldots, M \\ 0 & \text{otherwise} \end{cases}$$

$$[A_0]_{ij} = \begin{cases} a\bar{b} & j = 0 \qquad i = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$[B_0]_{ij} = \begin{cases} \bar{a}b & j = Q + i & i = 0, 1, 2, \ldots, s \\ ab & j = Q + i - 1 & i = 0, 1, 2, \ldots, s \\ \bar{a}\bar{b} & j = i & i = 0, 1, 2, \ldots, s \\ a\bar{b} & j = i - 1 & i = 1, 2, \ldots, s \\ \bar{a} & j = i & i = s + 1, s + 2, \ldots, S \\ a & j = i - 1 & i = s + 1, s + 2, \ldots, S \\ 0 & \text{otherwise} \end{cases}$$

For $k = 1, 2, \ldots, M$,

$$[B_k]_{ij} = \begin{cases} \bar{a} b r_1 r_2^k & j = Q + i \quad i = 0, 1, 2, \ldots, s \\ a b r_1 r_2^k & j = Q + i - 1 \quad i = 0, 1, 2, \ldots, s \\ (1 - \delta_{k,M}) \bar{a} \bar{b} + \delta_{k,M} \bar{b} & j = 0 \quad i = 0 \\ a \bar{b} & j = 0 \quad i = 1 \\ \bar{a} \bar{b} r_1 r_2^k & j = i \quad i = 1, 2, \ldots, s \\ a \bar{b} r_1 r_2^k & j = i - 1 \quad i = 2, 3, \ldots, s \\ \bar{a} r_1 r_2^k & j = i \quad i = s + 1, \ldots, S \\ a r_1 r_2^k & j = i - 1 \quad i = s + 1, \ldots, S \\ 0 & \text{otherwise} \end{cases}$$

$$[C_k]_{ij} = \begin{cases} \bar{a} b (1 - r_1 r_2^k) & j = Q + i - 1 \quad i = 0, 1, 2, \ldots, s \\ a b (1 - r_1 r_2^k) & j = Q + i - 2 \quad i = 0, 1, 2, \ldots, s \\ \bar{a} \bar{b} (1 - r_1 r_2^k) & j = i - 1 \quad i = 1, 2, \ldots, s \\ a \bar{b} (1 - r_1 r_2^k) & j = i - 2 \quad i = 2, 3, \ldots, s \\ \bar{a} (1 - r_1 r_2^k) & j = i - 1 \quad i = s + 1, s + 2, \ldots, S \\ a (1 - r_1 r_2^k) & j = i - 2 \quad i = s + 1, s + 2, \ldots, S \\ 0 & \text{otherwise} \end{cases}$$

Here the matrices A_0, B_k and C_k are the square matrices of order $(S + 1)$.

4. Calculating limiting probabilities

It can be seen from the structure of P, the homogeneous Markov chain $\{(X_n, L_n),\ n \in N\}$ on the finite state space is irreducible. Hence the limiting probability distribution

$$\pi^{(i,j)} = \lim_{n\to\infty} \Pr[X_n = i, L_n = j | X_0 = k, L_0 = l]$$

where $\pi^{(i,j)}$ is the steady state probability for the state (i, j) exists and is independent of the initial state (k, l). Let Π be the steady state limiting probability vector of P. That is, Π satisfies $\Pi P = \Pi$ and $\Pi e = 1$.

The vector Π can be represented by $\Pi = (\Pi^{<0>}, \Pi^{<1>}, \ldots, \Pi^{<M>})$ and $\Pi^{<i>} = (\pi^{(i,0)}, \pi^{(i,1)}, \pi^{(i,2)} \ldots, \pi^{(i,S)})$, for $i = 0, 1, 2, \ldots, M$.

Now the structure of P shows, the model under study is a finite birth death model in the Markovian environment. Hence we use the algorithm discussed by Gaver et al. [9] for computing the limiting probability vector. For the sake of completeness we provide the algorithm here.

Algorithm:

1. Determine recursively the matrix D_i, $0 \le i \le M$ by using

$D_0 = B_0$ and $D_i = B_i + C_i(I - D_i - 1)^{-1} A_0, i = 1, 2, \ldots, M$.

2. Solve the system $\Pi^{<M>}(I - D_M) = 0$
3. Compute recursively the vector $\Pi^{<i>}$, using $\Pi^{<i>} = \Pi^{<i+1>} C_i + 1(I - D_i)^{-1}$, $i = M - 1, M - 2, \ldots, 0$
4. Normalize the vector Π, by using $\Pi e = 1$.

5. System performance measures

In this section, we numerically illustrate the main performance measures of the model. First we provide expression for few system performance measures.

5.1. Mean inventory level

Let X_i denote the expected inventory level in the steady state. Since $\pi^{<i,j>}$ is the steady state probability vector when the number of demands in the orbit is i and the inventory level is j. Hence the expected inventory level is given by

$$X_i = \sum_{i=0}^{M} \sum_{j=1}^{S} j \pi^{(i,j)}$$

5.2. Expected reorder rate

To compute the mean reorder rate X_r, we consider the event of triggering a reorder' which occurs when the inventory level drops to s or less than that. Since a drop occurs from $s + 1$ with primary demand and/or the retrial demand and a drop from $s + 2$ with both primary demand and a retrial demand, we get

$$X_r = a\pi^{(0,s+1)} + \sum_{k=1}^{M} \left(a r_1 r_2^k + 1 - r_1 r_2^k\right) \pi^{(k,s+1)} + \sum_{k=1}^{M} \left(a(1 - r_1 r_2^k\right) \pi^{(k,s+2)}$$

5.3. Expected number of demands in the orbit

Let X_o denote the expected number of demands in the orbit in the steady state. Since $\Pi^{<i>}$ is the steady state probability vector when the number of demands in the orbit is i, with each component specifying a particular combination of the inventory level, $\Pi^{<i>}\mathbf{e}$ gives the probability that the number of demands in the orbit is i in the steady state. Hence the expected number of demands in the orbit is given by

$$X_o = \sum_{k=0}^{M} \Pi^{<k>}\mathbf{e}$$

5.4. Probability for the demand lost

Let X_b denote the probability for the demand lost. The demand occurs at the stock out period with full retrial orbit are considered to be lost. The probability for that event is defined as,

$$X_b = a\pi^{(M,0)}$$

5.5. Fraction of successful rate of retrial

Let X_{FR} denote the fraction of successful rate of retrial and is defined as,

$$X_{FR} = \frac{\sum_{i=1}^{M}\sum_{j=2}^{S}\left(1 - r_1 r_2^k\right)\pi^{(i,j)} + \sum_{i=1}^{M}\bar{a}\left(1 - r_1 r_2^k\right)\pi^{(i,1)}}{\sum_{i=1}^{M}\sum_{j=0}^{S}\left(1 - r_1 r_2^k\right)\pi^{(i,j)}}$$

6. Numerical Illustration

The long run total expected cost rate is defined as,

$$TC(S, s) = c_h X_i + c_s X_r + c_w X_o + c_l X_b$$

Here, c_s, c_h, c_w and c_l denote respectively, setup cost for the order, inventory carrying cost, waiting cost for the pending demand and cost for demand lost.

Due to the complex form of the limiting distribution, it is difficult to discuss the properties of the cost function analytically. Although we have not established analytically, our experience with considerable numerical examples indicates the function, $TC(S, s)$ to be convex.

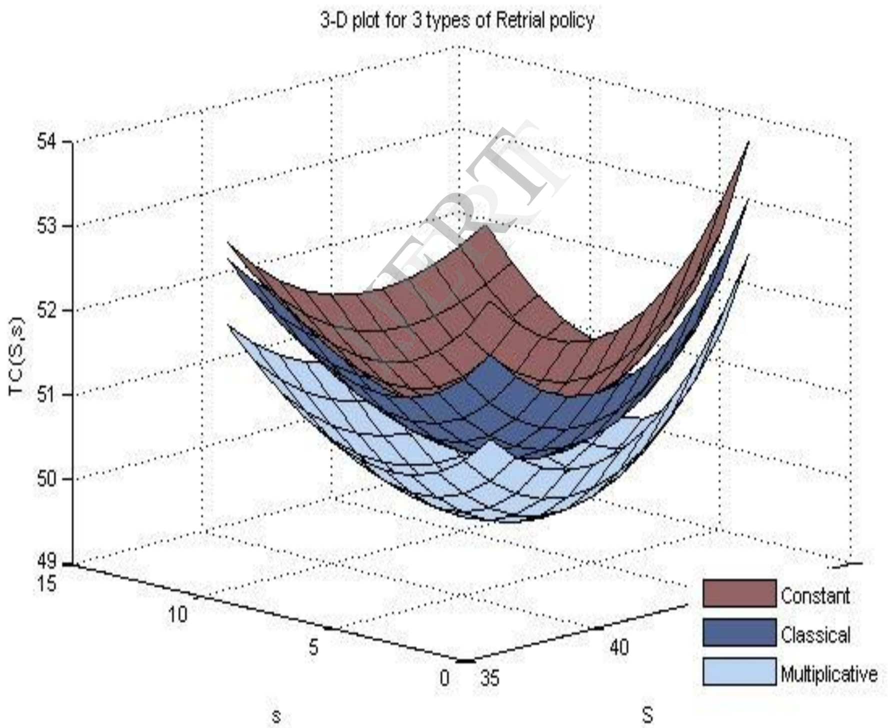

Figure 1: A three dimensional plot of the cost function TC(S, s)

A typical 3-dimensional plot of $TC(S, s)$ is presented in Figure 1. Here we show multiplicative retrial policy is the effective one, when we compare with the other two retrial policies.

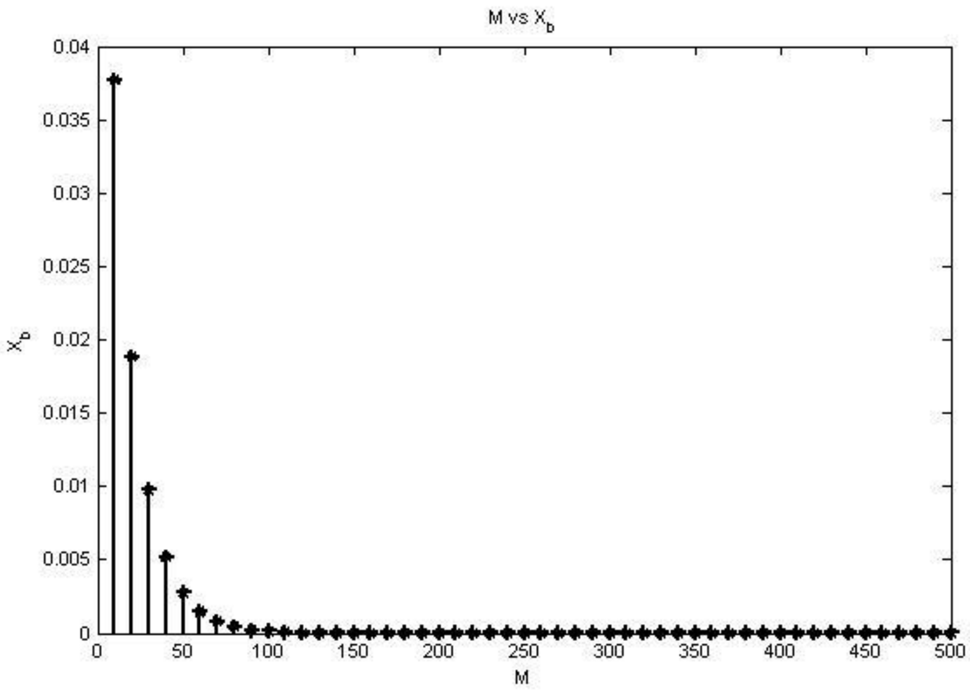

Figure 2: 2-dimensional plot for M versus Probability for demand lost

In Figure 2, we present the 2-dimensional plot for X_b versus M. Here we show the probability for the demand to be lost closer to zero when the orbit size becomes large. From Table 1 and 2 the total expected cost rate increases when c_s, c_h, c_l and c_w increase. From Table 3, the fraction of successful rate of retrial increases when the retrial probability decreases.

Table 1: c_h vs c_w on $TC(s,S)$

	c_w=2	c_w=3	c_w=4	c_w=5	c_w=6
c_h=0.1	11.496419	16.364520	21.232622	26.100723	30.968825
c_h=0.2	13.010480	17.878582	22.746683	27.614785	32.482886
c_h=0.3	14.524542	19.392643	24.260745	29.128846	33.996948
c_h=0.4	16.038603	20.906705	25.774807	30.642908	35.511010
c_h=0.5	17.552665	22.420767	27.288868	32.156970	37.025071

Table 2: c_s vs c_l on $TC(s,S)$

	c_l=1	c_l=2	c_l=3	c_l=4	c_l=5
c_s=5	19.197772	19.197772	19.197772	19.197772	19.197773
c_s=10	19.249054	19.249054	19.249054	19.249054	19.249055
c_s=15	19.300336	19.300336	19.300336	19.300336	19.300337
c_s=20	19.351618	19.351618	19.351618	19.351619	19.351619
c_s=25	19.402900	19.402900	19.402900	19.402901	19.402901

Table 3: Variation of r_1 & r_2 on X_{FR}

	r_2=0.2	r_2=0.4	r_2=0.6	r_2=0.8	r_2=1.0
r_1=0.2	0.277441	0.278487	0.280448	0.285471	0.326453
r_1=0.4	0.278093	0.280199	0.284187	0.294627	0.395160
r_1=0.6	0.278748	0.281928	0.288011	0.304287	0.496373
r_1=0.8	0.279405	0.283676	0.291919	0.314472	0.659717
r_1=1.0	0.280064	0.285440	0.295908	0.325145	-

7. Conclusion

In this paper we considered a discrete time inventory system with demands occurring according to a Bernoulli process. The inventory is replenished according to (s, S) ordering policy and we assume lead times are geometric distribution. The demands that occur during stock out period is permitted enter into the orbit of finite size. These orbiting demands retry to get their requirement after some random time and the time interval between two consecutive requests is distributed as geometric and the retrial policy is the multiplicative retrial policy. We assume the demands that occur during the stock-out period with full retrial orbit are considered to be lost. So we fix the maximum orbit size so that the probability for the demand last is negligible. We derived the joint probability distribution of the number of demands in the orbit and the inventory level is obtained in steady state case. We studied the effect of varying the cost and the other system parameters on the optimal values and the results agreed with what one would expect.

8. References

[1] Abboud, N. E., (2001), A discrete-time Markov production-inventory model with machine breakdowns, Computers & Industrial Engineering, 39, 95 - 107.

[2] Alfa, A. S., (2002), Discrete Time Queues and Matrix-Analytic Methods, Top, 10(2), 147-185.

[3] Alfa, A. S. (2010), Queueing Theory for Telecommunications, Springer, New York Dordrecht Heidelberg London.

[4] Artalejo, J. R., Gomez-Corral,A., (1997), Steady State Solution of a Single-Server Queue with Linear Repeated Requests, Journal of Applied Probability, 34(1), 223-233

[5] Artalejo, J. R., (2010), Accessible bibliography on retrial queues: Progress in 2000-2009, Mathematical and Computer Modelling 51, 1071 - 1081.

[6] Atencia, I., Moreno, P., (2008), A discrete-time retrial queue with multiplicative repeated attempts, J Appl Math Comput, 27, 63 - 75.

[7] Bar-Lev, S. K. and Perry, D. (1989), A discrete time Markovian inventory model for perishable commodities, Stochastic Analysis and Applications, 7, 243 - 259.

[8] Baumann, H., Sandmann, W., (2010), Numerical solution of level dependent quasi-birth-and-death processes, Procedia Computer Science, 1, 1555 - 1563.

[9] Gaver, D.P., Jacobs, P.A. and Latouche, G., (1984),Finite birth-and-death models in randomly changing environments, Advances in Applied Probability, 16, 715 - 731

[10] Lian, Z. and Liu, L., (1999), A discrete-time model for perishable inventory systems, Annals of Operations Research, 87, 103 - 116.

[11] Lian, Z., Liu, L. and Neuts, M. F., (2005), A discrete-time model for common lifetime inventory systems, Mathematics of Operations Research, 30(3), 718 - 732.