A Discrete Time Inventory System With Retrial Demands

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Abstract
In this paper, a discrete time inventory system with demands occurring according to a Bernoulli process. The inventory is replenished according to an \((s,S)\) policy and the lead times are assumed to follow a geometric distribution. The demands that occur during stock out period is permitted enter into the orbit of finite size. These orbiting demands retry to get their requirement after some random time and the time interval between two consecutive requests is distributed as geometric and the retrial policy is the multiplicative retrial policy. The joint probability distribution of the number of demands in the orbit and the inventory level is obtained in steady state case. Some system performance measures are derived and the results are illustrated numerically.

Keywords: Discrete time inventory system, \((s,S)\) policy, Retrial demand, Multiplicative retrial policy

1. Introduction
Inventory models have been considered under continuous review as well as periodic review. In the recent past discrete time models have started receiving attention of researchers in the areas of queuing and telecommunications [2, 3]. In discrete time setting, it is assumed that the time axis is calibrated into epochs by small units and that all the events are deemed to occur only at these epochs. With the advent of fast computing devices and efficient transaction reporting facilities, such epochs with small gaps can be conveniently assumed so that events can occur at these epochs.

In the case of inventory modelling under discrete times, the first paper was by Bar-Lev and Perry [7], who assumed that demands are non-negative integer value random variables and items have constant life times. Lian and Liu [10] developed a discrete time inventory model with geometrically distributed inter-demand times, bulk demands and constant life time for items. They assumed \((0,S)\) ordering policy, with instantaneous supply which clears any backlog and restores the stock to the maximum capacity \(S\). This assumption helped them to have fixed life time for all items. They derived the limiting distribution of inventory level through matrix-analytic method.

Abboud [1] studied a discrete inventory model for production inventory systems with machine breakdowns. They assumed that the demand and production rates were constant and that the failure and repair times of each item were independently distributed as geometric. Lian et al. [11] developed a discrete time inventory system with discrete PH-renewal process for (batch) demand time points and assumed discrete- PH-distribution for life time of items. They also assumed zero lead time and that unsatisfied demand were completely backlogged.

I. Atentia and P. Moreno [6] developed a discrete time retrial queueing system with multiplicative repeated attempts. The authors used the retrial policy is the multiplicative one which analogous to the linear retrial policy in continuous-time [4].

The rest of the paper is organized as follows. In Section 2, we describe the problem under consideration. The mathematical model of our problem is presented in section 3. The limiting probability vector of the chain is calculated in Section 4. In section 5 we derived some important system performance measures and in the final section the expression for total expected cost rate and we provide few numerical illustrations of the results.

- \(e\) : a column vector of \(1\)’s with appropriate dimension
- \(\delta_{ij}\): Kronecker delta function

2. Problem formulation
We consider a discrete time retrial inventory system where the time axis is divided into intervals of equal length, called slots (epochs). It is assumed that all system activities (arrivals, retrials and replenishment) occur at the slot boundaries, and
therefore they may occur at the same time. The maximum capacity of the inventory is $S$.

- Demand arrive according to a Bernoulli process with probability $a$, thus $a$ is the probability that a demand arrives at a slot and $\bar{a} (= 1 - a)$, is the probability that an arrival does not take place in a slot. When the on-hand inventory level is more than one then the arriving demand is satisfied immediately.

- We adopt $(s, S)$ ordering policy, that is, when the number of available items reaches the value $s$, an order for $Q(s + S - s > s)$ items is placed which is delivered after a lead time of geometric distribution with parameter $c > 0$. The condition $Q > S$ is assumed so that when the supply of an order is received during the stock out period, the inventory level would be brought above the reorder level.

- If the arrival finds the inventory level zero, he enters into the orbit of pending demands and he retry after a random amount of time. In order to overcome analytical difficulties, we assume the most natural approach of restricting the retrial group to be of finite size. The arrivals of demands at the time of empty stock with full retrial orbit are assumed to be lost. So, in what follows, $M$ will denote the maximum number of simultaneous pending demand in which the probability for the demand to be lost is negligible. If more than one repeated demand retries at the same slot, any of them is randomly selected and the others must go back to the retrial group. The time between two successive repeated attempts is geometrically distributed with probability $1 - r_1^{-\bar{a} k} r_2^k (r_1, r_2 \in [0, 1], r_1 r_2 \neq 1)$ given that there are $k$ demands in the orbit. If $r_2 = 1$, the retrial policy becomes the constant retrial policy, when $r_1 = 1$, we get the classical retrial policy.

Unlike continuous review inventory systems, multiple events such as demand, supply and retrial from the orbit may occur between epochs $n$ and $n + 1, n = 0, 1, 2, \ldots$. Hence we adopt the following convention: If the events such as demand for an item, retrial from the orbit and supply of an order takes place at $n$ ($n = 1, 2, 3, \ldots$), it is assumed that first supply is received then demand occurs and finally retrial from the orbit takes place.

3. Model Descriptions

Let $X_n$ and $L_n$ denote respectively the number of demands in the orbit and the inventory level at time $n$. From the assumptions made on the input and output processes, it can be shown that the stochastic process $(X_n, L_n), n \in N$ is a Discrete Time Markov Chain with state space given by $E = \{(i, j) : i = 0, 1, \ldots, M, j = 0, 1, \ldots, S\}$.

The transition probability function is defined as,

$$P((i, j), (k, l)) = \Pr[X_{n+1} = k, L_{n+1} = l | X_n = i, L_n = j, (i, j), (k, l) \in E]$$

The transition probability matrix $P$ of this process, $P = \{(P((i, j), (k, l))) ((i, j), (k, l)) \in E$ is given by,

$$[P]_{il} = \begin{cases} A_0 & l = k + 1 \quad k = 0, 1, \ldots, M - 1 \\ B_k & l = k \quad k = 0, 1, \ldots, M \\ C_k & l = k - 1 \quad k = 1, 2, \ldots, M \\ 0 & \text{otherwise} \end{cases}$$

$$[A_0]_{ij} = \begin{cases} ab & j = 0 \quad i = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$[B_0]_{ij} = \begin{cases} ab & j = Q + i \quad i = 0, 1, 2, \ldots, s \\ ab & j = Q + i - 1 \quad i = 0, 1, 2, \ldots, s \\ \bar{a} & j = i - 1 \quad i = 1, 2, \ldots, s \\ a & j = i \quad i = s + 1, s + 2, \ldots, S \\ 0 & \text{otherwise} \end{cases}$$
For $k = 1, 2, \ldots, M$,

\[
[B_k]_{ij} = \begin{cases} 
\hat{a} b r_{1} k_{j} & j = Q + i - 1 \quad i = 0, 1, 2, \ldots, s \\
\hat{a} b r_{1} k_{j} & j = Q + i - 2 \quad i = 0, 1, 2, \ldots, s \\
\hat{a} b r_{1} k_{j} & j = i - 1 \quad i = 1, 2, \ldots, s \\
\pi & j = i \quad i = s + 1, \ldots, S \\
\pi & j = Q + i - 1 \quad i = 0, 1, 2, \ldots, s \\
\pi & j = 0 \quad i = 1 \\
\pi & j = 0 \quad i = 1 \\
\pi & j = i - 1 \quad i = s + 1, \ldots, S \\
\pi & j = i \quad i = s + 1, \ldots, S \\
0 & \text{otherwise} 
\end{cases}
\]

Here the matrices $A_0, B_k$ and $G_k$ are the square matrices of order $(S + 1)$.

4. Calculating limiting probabilities

It can be seen from the structure of $P$, the homogeneous Markov chain $(X_n, l_n), n \in \mathbb{N}$ on the finite state space is irreducible. Hence the limiting probability distribution

\[
\pi^{(ij)} = \lim_{n \to \infty} \text{Pr}[X_n = i, L_n = j|X_0 = k, L_0 = l]
\]

where $\pi^{(ij)}$ is the steady state probability for the state $(i, j)$ exists and is independent of the initial state $(k, l)$. Let $\Pi$ be the steady state limiting probability vector of $P$. That is, $\Pi$ satisfies $\Pi P = \Pi$ and $\Pi e = 1$.

The vector $\Pi$ can be represented by $\Pi = (\Pi^{<0>}, \Pi^{<1>}, \ldots, \Pi^{<M>})$ and $\Pi^{<i>} = (\pi^{(0,i)}, \pi^{(1,i)}, \ldots, \pi^{(S,i)})$, for $i = 0, 1, 2, \ldots, M$.

Now the structure of $P$ shows, the model under study is a finite birth death model in the Markovian environment. Hence we use the algorithm discussed by Gaver et al. [9] for computing the limiting probability vector. For the sake of completeness we provide the algorithm here.

**Algorithm:**

1. Determine recursively the matrix $D_i$, $0 \leq i \leq M$ by using

\[
D_0 = B_0 \quad \text{and} \quad D_i = B_i + C_i (1 - D_i - 1)^{-1} A_i, \quad i = 1, 2, \ldots, M.
\]

2. Solve the system $\Pi^{<i>}(1 - D_i) = 0$

3. Compute recursively the vector $\Pi^{<i>}$

\[
\Pi^{<i>} = \Pi^{<i+1>} C_i + (1 - D_i)^{-1}, \quad i = M - 1, M - 2, \ldots, 0
\]

4. Normalize the vector $\Pi$, by using $\Pi e = 1$.

5. System performance measures

In this section, we numerically illustrate the main performance measures of the model. We provide expression for few system performance measures.

5.1. Mean inventory level

Let $X_i$ denote the expected inventory level in the steady state. Since $\pi^{(i,j)}$ is the steady state probability vector when the number of demands in the orbit is $i$ and the inventory level is $j$. Hence the expected inventory level is given by

\[
X_i = \sum_{j=0}^{M} j \pi^{(i,j)}
\]

5.2. Expected reorder rate

To compute the mean reorder rate $X_r$, we consider the event of triggering a reorder which occurs when the inventory level drops to $s$ or less than that. Since a drop occurs from $s + 1$ with primary demand and/or the retrial demand and a drop from $s + 2$ with both primary demand and a retrial demand, we get

\[
X_r = a n^{(0,s+1)} + \sum_{k=1}^{M} \left[ a r_1 r_2^k + 1 - r_1 r_2^k \right] \pi^{(k,s+1)} + \sum_{k=1}^{M} a (1 - r_1 r_2^k) \pi^{(k,s+2)}
\]

5.3. Expected number of demands in the orbit

Let $X_0$ denote the expected number of demands in the orbit in the steady state. Since $\pi^{<i>}$ is the steady state probability vector when the number of demands in the orbit is $i$, with each component
specifying a particular combination of the inventory level, $\Pi_{<i>e}$ gives the probability that the number of demands in the orbit is $i$ in the steady state. Hence the expected number of demands in the orbit is given by

$$X_o = \sum_{k=0}^{M} \Pi_{<k>e}$$

### 5.4. Probability for the demand lost

Let $X_{b}$ denote the probability for the demand lost. The demand occurs at the stock out period with full retrial orbit are considered to be lost. The probability for that event is defined as,

$$X_{b} = \pi(M,0)$$

### 5.5. Fraction of successful rate of retrial

Let $X_{FR}$ denote the fraction of successful rate of retrial and is defined as,

$$X_{FR} = \frac{\sum_{i=1}^{M} \sum_{j=0}^{S} (1-r_1 r_2^j) \pi(i,j) + \sum_{i=1}^{M} \sum_{j=0}^{S} \bar{a}(1-r_1 r_2^j) \pi(i,1)}{\sum_{i=1}^{M} \sum_{j=0}^{S} (1-r_1 r_2^j) \pi(i,j)}$$

### 6. Numerical Illustration

The long run total expected cost rate is defined as,

$$TC(S,s) = c_h X_i + c_r X_r + c_w X_o + c_l X_b$$

Here, $c_h, c_r, c_w$ and $c_l$ denote respectively, setup cost for the order, inventory carrying cost, waiting cost for the pending demand and cost for demand lost.

Due to the complex form of the limiting distribution, it is difficult to discuss the properties of the cost function analytically. Although we have not established analytically, our experience with considerable numerical examples indicates the function, $TC(S,s)$ to be convex.

![Figure 1: A three dimensional plot of the cost function $TC(S,s)$](image)

A typical 3-dimensional plot of $TC(S,s)$ is presented in Figure 1. Here we show multiplicative retrial policy is the effective one, when we compare with the other two retrial policies.
In Figure 2, we present the 2-dimensional plot for $X_b$ versus M. Here we show the probability for the demand to be lost closer to zero when the orbit size becomes large. From Table 1 and 2 the total expected cost rate increases when $c_s, c_h, c_l$ and $c_w$ increase. From Table 3, the fraction of successful rate of retrial increases when the retrial probability decreases.

**Table 1: $c_h$ vs $c_w$ on $TC(s,S)$**

<table>
<thead>
<tr>
<th>$c_h$</th>
<th>$c_w=2$</th>
<th>$c_w=3$</th>
<th>$c_w=4$</th>
<th>$c_w=5$</th>
<th>$c_w=6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>11.496419</td>
<td>16.364520</td>
<td>21.232622</td>
<td>26.100723</td>
<td>30.968825</td>
</tr>
<tr>
<td>0.2</td>
<td>13.010480</td>
<td>17.878582</td>
<td>22.746683</td>
<td>27.614785</td>
<td>32.482886</td>
</tr>
<tr>
<td>0.3</td>
<td>14.524542</td>
<td>19.392643</td>
<td>24.260745</td>
<td>29.128846</td>
<td>33.996948</td>
</tr>
<tr>
<td>0.4</td>
<td>16.038603</td>
<td>20.906705</td>
<td>25.774807</td>
<td>30.642908</td>
<td>35.511010</td>
</tr>
<tr>
<td>0.5</td>
<td>17.552665</td>
<td>22.420767</td>
<td>27.288868</td>
<td>32.156970</td>
<td>37.025071</td>
</tr>
</tbody>
</table>

**Table 2: $c_s$ vs $c_l$ on $TC(s,S)$**

<table>
<thead>
<tr>
<th>$c_s$</th>
<th>$c_l=1$</th>
<th>$c_l=2$</th>
<th>$c_l=3$</th>
<th>$c_l=4$</th>
<th>$c_l=5$</th>
</tr>
</thead>
</table>
Table 3: Variation of $r_1$ & $r_2$ on $X_{FR}

<table>
<thead>
<tr>
<th>$r_1$</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1.0</th>
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</thead>
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<td>$r_2=0.2$</td>
<td>0.277441</td>
<td>0.278093</td>
<td>0.278748</td>
<td>0.279405</td>
<td>0.280064</td>
</tr>
<tr>
<td>$r_2=0.4$</td>
<td>0.278487</td>
<td>0.280199</td>
<td>0.281928</td>
<td>0.283676</td>
<td>0.285440</td>
</tr>
<tr>
<td>$r_2=0.6$</td>
<td>0.280448</td>
<td>0.284187</td>
<td>0.288011</td>
<td>0.288748</td>
<td>0.291919</td>
</tr>
<tr>
<td>$r_2=0.8$</td>
<td>0.285471</td>
<td>0.294627</td>
<td>0.304287</td>
<td>0.308748</td>
<td>0.314472</td>
</tr>
<tr>
<td>$r_2=1.0$</td>
<td>0.326453</td>
<td>0.395160</td>
<td>0.496373</td>
<td>0.597487</td>
<td>0.659717</td>
</tr>
</tbody>
</table>

7. Conclusion
In this paper we considered a discrete time inventory system with demands occurring according to a Bernoulli process. The inventory is replenished according to ($s, S$) ordering policy and we assume lead times are geometric distribution. The demands that occur during stock out period is permitted enter into the orbit of finite size. These orbiting demands retry to get their requirement after some random time and the time interval between two consecutive requests is distributed as geometric and the retrial policy is the multiplicative retrial policy. We assume the demands that occur during the stock-out period with full retrial orbit are considered to be lost. So we fix the maximum orbit size so that the probability for the demand last is negligible. We derived the joint probability distribution of the number of demands in the orbit and the inventory level is obtained in steady state case. We studied the effect of varying the cost and the other system parameters on the optimal values and the results agreed with what one would expect.

8. References